# Exact Spectrum and Time-Domain Output of Flying-Adder Frequency Synthesizers 

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#### Abstract

The spectrum and time-domain output of the fly-ing-adder frequency synthesizer are derived analytically. The amplitude and phase of the average-frequency component are derived in closed forms. The theoretical results are verified by spectral measurements of an FPGA implementation and by numerical simulation.


## I. Introduction

THE flying-adder (FA) frequency synthesizer was introduced to generate the clock signal for digital circuits [1], [2]. It has been treated in several publications [3]-[14], where it is also referred to as direct digital period synthesizer (DDPS) or digital-to-frequency converter (DFC). A rigorous mathematical theory of the operation of the flying-adder frequency synthesizer has been developed in [15] and [16].

The FA is shown in simplified form in Fig. 1, driven by a multiphase clock like that in Fig. 2. It shares some building blocks with direct digital synthesizers (DDS) [17], [18] as well as other phase-switching prescalers and similar structures [19]-[25].

There is a significant structural difference between the FA and DDS. In DDS, the value of the phase accumulator is updated at the fixed reference clock rate, in contrast to FA which has a feedback path updating its register at a variable rate (for most parameters' values). This results in a linear control of the period of the FA versus linear control of the frequency in the DDS.

The FA offers simplicity, good period resolution, wide output frequency range and design convenience (it is fully digital) and has been used successfully in driving digital circuits [11]. It suffers, however, from certain bounded timing irregularity of the pulses [16] which results in a rich spurious frequency content. Efforts have been devoted to estimate the spectrum of the FA [26]-[28] resulting in algorithmic approaches.

This work provides a rigorous mathematical derivation of exact analytic expressions of FA's spectrum and output signal $v(t)$, as well as compact closed-form expressions of the amplitude and phase of FA's average-frequency component [15]. This work responds to the request for theoretical analysis of FA's operation in [29] and aims to lead to a better understanding of FA's spectral properties and potential application of FA in analog and RF systems.

[^0]In the process of deriving the spectrum, the output of the FA is expressed as a uniformly sampled-and-held ideal $50 \%$ duty cycle squarewave, providing a simple equivalent representation of FA's signal processing operation.

Section II of the paper is a brief introduction to FA's operation and the timing of the output signal $v(t)$ which is used in Section III to express $v(t)$ as a uniformly sampled ideal squarewave. Section IV derives analytically FA's output in the frequency-domain, converts it to time-domain, derives the frequencies of the frequency components and calculates the amplitude and phase of average-frequency component. Finally, Section V compares the theoretical results with measurements based on an FPGA implementation of the FA.

## II. Notation and Operation of the FA

The basic structure of the FA is shown in Fig. 1. It is composed of the following.

An $n$-bit register, of value $x_{k} \in\left\{0,1, \ldots, 2^{n}-1\right\}$, triggered by the rising edges of signal $s(t)$. Index $k$, of $x_{k}$, is our discrete-time reference and is formally defined as the function of time:
$k(t)=$ number of rising edges (or spikes) of signal $s(t)$ within the time interval $\left(0^{-}, t^{+}\right)$.

Without loss of generality, we assume that the initial value of the register is $x_{0}=0$, for $\mathrm{t}<0$.

- A truncation block that keeps only the first $m$ most significant bits (MSB) of the register resulting in $y_{k}$ $\in\left\{0,1, \ldots, 2^{m}-1\right\}$. In this paper we assume that $1<$ $m<n$. Note that if $m=n$, the FA operates simply as an integer frequency divider [16].
- A multiplexer (MUX) driven by a family of $2^{m}$ phases, $\Phi_{0}(t), \Phi_{1}(t), \ldots, \Phi_{2^{m}-1}(t)$ of a periodic $50 \%$ duty-cycle square-wave. The phases are uniformly shifted, i.e., $\Phi_{k}(t)=\Phi_{0}(t-k \Delta)$ for $k=1,2, \ldots, 2^{m}-1$, where $\Delta$ $=T / 2^{m}=1 /\left(2^{m} f_{\text {clk }}\right)$, like those in Fig. 2, typically generated by a ring oscillator. The MUX selects the $y_{k}$ th phase, i.e., its output is $s(t)=\Phi_{y_{k(t)}}(t)$.
- An adder that adds the frequency control word $w$ to the value of the register $x_{k}$. In this paper we assume that $w \geq 2^{n-m}$. It can be shown that for most values of $w<2^{n-m}$, the output of the FA is a very irregular waveform perhaps with limited applications [15].
- The D-Flip-Flop (D-FF) counting the rising edges (or spikes) of $s(t)$ modulo 2 . Without loss of generality, we assume that the initial value of the D-FF, at $k=$ 0 , is $Q=0$.


Fig. 1. Flying adder (shown with 4 input phases, $\Phi_{0}$ to $\Phi_{3}$ ).

From these, we can conclude the following equations describing the operation of the FA:

$$
\begin{equation*}
x_{k}=(k w) \bmod 2^{n} \tag{1}
\end{equation*}
$$

as well as

$$
\begin{equation*}
y_{k}=x_{k} \operatorname{div} 2^{n-m}=\left[\frac{x_{k}}{2^{n-m}}\right] \tag{2}
\end{equation*}
$$

for $k=0,1,2, \ldots$, where $[a]$ is the integer part of real number $a$. Eq. (2) can also be written as

$$
\begin{equation*}
y_{k}=\left[\frac{k w}{2^{n-m}}\right] \bmod 2^{m} \tag{3}
\end{equation*}
$$

using ${ }^{1}\left(a \bmod 2^{c}\right) \operatorname{div} 2^{d}=\left(a \operatorname{div} 2^{d}\right) \bmod 2^{c-d}$.
To illustrate the timing of the signals in the FA, let's consider Fig. 3, presenting the MATLAB (The MathWorks, Natick, MA) simulation results of the FA in Fig. 1 with parameters $n=4, m=2$, and frequency word $w$ $=7$.

The $x$-axis in the graph is the (real) continuous time, $t$, measured in multiples of $\Delta=T / 2^{m}$. The four input phases, $\Phi_{1}, \Phi_{2}, \Phi_{3}$, and $\Phi_{4}$ are shown in Fig. 3(a). The time intervals in which each phase is selected by the MUX are indicated with thick line segments.

The values of parameters $n, m$, and $w$ used in this example result in signal $s(t)$, shown in Fig. 3(e), that is composed only of spikes ${ }^{2}$ (in this example). By definition, dis-crete-time $k$, shown in part Fig. 3(f), results from counting

[^1]

Fig. 2. The $2^{m}$ input phases to the FA, typically generated by a ring oscillator.
the rising edges of $s$. Discrete-time has value $k$ between the $k$ th and the $k+1$ rising edges of $s$.

Figs. 3(b)-(d) present the continuous-time versions ${ }^{3}$ of the discrete-time sequences $\left\{x_{k}\right\},\left\{y_{k}\right\}$, and $\left\{\delta_{k}\right\}$, with the positive integer sequence $\left\{\delta_{k}\right\}$ defined as

$$
\delta_{k}= \begin{cases}\left(y_{k}-y_{k-1}\right) \bmod 2^{m} & \text { if } y_{k} \neq y_{k-1}  \tag{4}\\ 2^{m} & \text { otherwise }\end{cases}
$$

Fig. 3 shows that if $y_{k-1} \neq y_{k}$, the time moment the $y$ register changes value from $y_{k-1}$ to $y_{k}$, the $k$ th rising edge (or spike) of $s(t)$ appears; moreover the $k+1$ rising edge appears $\left(\left(y_{k}-y_{k-1}\right) \bmod 2^{m}\right) \Delta$ seconds after the $k$ th one because of the time-offset between the clock phases. In this example, the 2 nd rising edge (a spike) of $s(t)$ appears at $t=\Delta$, and, because $y_{1}=1$ and $y_{2}=3$, the 3rd rising edge appears at $t=3 \Delta$.

Consider now the case of the FA with $n=4, m=2$, and $w=14$. MATLAB simulation provides the FA's signals shown in Fig. 4. We observe that $s(t)$ is a sequence of pulses and that because $y_{2}=y_{1}$ the second rising edge of $s(t)$ at $t=3 \Delta$ does not result in a change of selected input phase. This implies that the next rising edge of $s(t)$ $=\Phi_{3}(t)$ appears $T=2^{m} \Delta$ seconds later (a full cycle), i.e., at $t=7 \Delta$.

The discussion leads to an important fact: the time interval between the $k$ th and $k+1$ rising edges (or spikes)

[^2]

Fig. 3. Flying adder, $n=4, m=2, w=7$.
of $s(t)$ is $\delta_{k} \Delta$ long. Moreover, the D-FF at the output of the FA counts the rising edges (or spikes) of $s(t)$ modulo 2, resulting in the output signal $v(t)$ shown in Figs. 3(g) and $4(\mathrm{~g})$.

From this discussion, we conclude that the rising and falling edges of the $j$ th pulse in the output signal $v(t)$ appear at

$$
\begin{equation*}
t_{j}=\Delta \sum_{k=1}^{2 j-2} \delta_{k} \tag{5}
\end{equation*}
$$

(we agree that $t_{1}=0$ ) and at

$$
\begin{equation*}
\tau_{j}=t_{j}+\Delta \delta_{2 j-1}, \tag{6}
\end{equation*}
$$

respectively, for all $j=1,2,3, \ldots$ It is shown in Lemma 1 in the Appendix that our assumptions of $0<m<n$ and $2^{n-m} \leq w<2^{n}$ lead to

$$
\begin{equation*}
t_{j}=\left[\frac{(2 j-2) w}{2^{n-m}}\right] \Delta \tag{7}
\end{equation*}
$$

and
(a)

(e)
(f)
(g)


Fig. 4. Flying adder, $n=4, m=2, w=14$.

$$
\begin{equation*}
\tau_{j}=\left[\frac{(2 j-1) w}{2^{n-m}}\right] \Delta \tag{8}
\end{equation*}
$$

for all $j=1,2,3, \ldots$.
Counting the pulses in $v(t)$ from $t=0^{-}$, the first pulse begins at $t=t_{1}=0$ and the $1+2^{n-m-1}$ pulse begins at $t$ $=t_{1+2^{n-m-1}}=w \Delta$. Therefore the time interval $\left[t_{1}, t_{1+2^{n-m-1}}\right)$ is composed of exactly $2^{n-m-1}$ complete cycles. Moreover, for every ${ }^{4} r \in \mathbb{Z}$ it is

$$
t_{j+2^{n-m-1} r}=t_{j}+r w \Delta
$$

and so function $v(t)$ has a period of $w \Delta$ seconds. We conclude that under our assumptions there are exactly $2^{n-m-1}$ cycles per period ${ }^{5}$ of $w \Delta$ seconds, implying that $v(t)$ has average frequency ${ }^{6}$

[^3]\[

$$
\begin{equation*}
f_{\mathrm{av}}=\frac{2^{n-m-1}}{w \Delta} \tag{9}
\end{equation*}
$$

\]

## III. Timing Details of the Output Signal

This section studies in more detail the timing of the output signal $v(t)$ and compares it to that of the ideal $50 \%$ duty-cycle periodic squarewave, of frequency $f_{\text {av }}$, formally defined as

$$
\begin{equation*}
\gamma(t)=\frac{1}{2}+\frac{1}{2} \operatorname{sgn}\left(\sin \left(2 \pi f_{\mathrm{av}} t\right)\right) \tag{10}
\end{equation*}
$$

where sgn is the signum function. Note that the rising and falling edges of the $j$ th pulse of this ideal periodic squarewave appear at

$$
\begin{equation*}
\tilde{t}_{j}=\frac{(2 j-2) w}{2^{n-m}} \Delta \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\tau}_{j}=\frac{(2 j-1) w}{2^{n-m}} \Delta \tag{12}
\end{equation*}
$$

respectively, for all $j=1,2,3, \ldots$.
To derive the spectrum of the FA analytically, we express the output signal $v(t)$ as a time-shifted version of $\gamma(t)$ sampled by the impulse sequence

$$
\begin{equation*}
\theta(t)=\sum_{r=-\infty}^{\infty} \delta(t-r \Delta) \tag{13}
\end{equation*}
$$

and convolved with the pulse $p(t)$ shown in Fig. 5.
To achieve our goal, we need to establish certain timing bounds. First we notice that the lengths of the 1-intervals and the 0 -intervals of the cycles in $v(t)$ are at least $\Delta$ seconds long as shown in Fig. 6. To derive this analytically we use (7) and (8), our assumption that $w \geq 2^{n-m}$, and the inequality $[a+b] \geq[a]+[b]$, which is valid for all real $a$ and $b$, to get

$$
\begin{equation*}
\tau_{j}-t_{j} \geq \Delta \tag{14}
\end{equation*}
$$

for the 1-intervals and

$$
\begin{equation*}
t_{j+1}-\tau_{j} \geq \Delta \tag{15}
\end{equation*}
$$

for the 0 -intervals.
Next, we relate the timing of the pulses in $v(t)$ to that of the pulses in $\gamma(t)$ to express $v(t)$ as a sampled-and-held version of $\gamma(t)$. From (7) and (11) we have

$$
\begin{align*}
\frac{\tilde{t}_{j}-t_{j}}{\Delta} & =\frac{(j-1) w}{2^{n-m-1}}-\left[\frac{(j-1) w}{2^{n-m-1}}\right] \\
& =\frac{\beta}{2^{n-m-1}} \tag{16}
\end{align*}
$$

where $\beta$ and $\alpha$ are the unique pair of integers satisfying both $(j-1) w=\alpha 2^{n-m-1}+\beta$ and $0 \leq \beta<2^{n-m-1}$. The


Fig. 5. Pulse function $p(t)$.
last inequality, along with (16), implies that $0 \leq \tilde{t}_{j}-t_{j} \leq$ $\Delta-\Delta / 2^{n-m-1}$, which gives

$$
\begin{equation*}
t_{j}+\frac{\Delta}{2^{n-m+1}} \leq \tilde{t}_{j}+\frac{\Delta}{2^{n-m+1}} \leq t_{j}+\Delta-\frac{\Delta}{2^{n-m+1}} \tag{17}
\end{equation*}
$$

Similarly, using (8) and (12) and the unique integers $\alpha^{\prime}$ and $\beta^{\prime}$ satisfying $(2 j-1) w=\alpha^{\prime} 2^{n-m}+\beta^{\prime}$ and $0 \leq \beta^{\prime}<$ $2^{n-m}$, we have

$$
\begin{aligned}
\frac{\tilde{\tau}_{j}-\tau_{j}}{\Delta} & =\frac{(2 j-1) w}{2^{n-m}}-\left[\frac{(2 j-1) w}{2^{n-m}}\right] \\
& =\frac{\beta^{\prime}}{2^{n-m}}
\end{aligned}
$$

which, along with the bounds of $\beta^{\prime}$, leads to

$$
\begin{equation*}
\tau_{j}+\frac{\Delta}{2^{n-m+1}} \leq \tilde{\tau}_{j}+\frac{\Delta}{2^{n-m+1}} \leq \tau_{j}+\Delta-\frac{\Delta}{2^{n-m+1}} \tag{18}
\end{equation*}
$$

Note that terms $\tilde{t}_{j}+\Delta / 2^{n-m+1}$ and $\tilde{\tau}_{j}+\Delta / 2^{n-m+1}$ in inequalities (17) and (18), respectively, are the rising and falling times of the $j$ th pulse of the time-shifted ideal periodic squarewave

$$
\begin{equation*}
\hat{\gamma}(t)=\gamma\left(t-\frac{\Delta}{2^{n-m+1}}\right) \tag{19}
\end{equation*}
$$

The timing of $v(t)$ and $\hat{\gamma}(t)$ is shown in Fig. 6, illustrating inequalities (14), (15), (17), and (18). Observe that the rising edge of $\hat{\gamma}(t)$ appears after the rising edge of $v(t)$ and less than $\Delta$ seconds after it. Moreover, the rising edge of $\hat{\gamma}(t)$ is at least $\Delta / 2^{n-m+1}$ seconds away ${ }^{7}$ from the impulses of $\theta(t)$. The situation is similar for the falling edges of $\hat{\gamma}(t)$.

Therefore, $v(t)$ can be written as the result of sampling $\hat{\gamma}(t)$ by the impulse sequence (13) and holding it by the pulse function $p(t)$ shown in Fig. 5, i.e.,

$$
\begin{equation*}
v(t)=(\hat{\gamma}(t) \theta(t)) * p(t) \tag{20}
\end{equation*}
$$

where $*$ stands for convolution.
Note that pulse $p(t)$ is not causal. This is not a problem here because we are not concerned with any implementation of the sample-and-hold process but rather with the mathematical expression of $v(t)$, given by (20). However, (20) implies that

[^4]

Fig. 6. The output signal $v(t)$ around its $j$ th pulse is shown as a solid line. The time-shifted ideal periodic squarewave $\hat{\gamma}(t)$ is shown as a dashed thick line. Bounds (14), (15), (17), and (18) imply that the $j$ th rising edge of $\hat{\gamma}(t)$ appears strictly within the first $\Delta$ sub-interval of the $j$ th 1 -interval (pulse) of $v(t)$ and that the $j$ th falling edge of $\hat{\gamma}(t)$ appears strictly within the first $\Delta$ sub-interval of the $j$ th 0 -interval of $v(t)$.

$$
\begin{equation*}
v(t-\Delta)=(\hat{\gamma}(t) \theta(t)) * p(t-\Delta) \tag{21}
\end{equation*}
$$

where $p(t-\Delta)$ is causal. The right side of (21) has a circuit equivalent discussed in the following section.

Expression (20) of $v(t)$ has the compact (almost) equivalent circuit interpretation shown in Fig. 7. Here the sample-and-hold process is performed by a rising-edgetriggered D-FF clocked by signal $h(t)$, instead of $\theta(t)$, and the output, $Q$, of the D-FF is $v(t-\Delta)$, instead of $v(t)$.

## IV. The Spectrum of FA

To calculate the spectrum of FA's output,

$$
\begin{equation*}
V(f)=\int_{-\infty}^{\infty} v(t) e^{-2 \pi i f t} d t \tag{22}
\end{equation*}
$$

Eq. (20) is used, implying that [30]

$$
\begin{equation*}
V(f)=(\hat{\Gamma} * \Theta)(f) \cdot P(f) \tag{23}
\end{equation*}
$$

where $\hat{\Gamma}$ and $\Theta$ are the Fourier transforms of $\hat{\gamma}$ and $\theta$, respectively. The Fourier transform of the ideal squarewave $\gamma(t)$ is derived from its Fourier series [30],

$$
\begin{equation*}
\gamma(t)=\frac{1}{2}+\frac{1}{\pi} \sum_{\ell: \text { odd }} \frac{1}{\ell} \sin \left(\frac{2^{n-m} \pi \ell}{w \Delta} t\right) \tag{24}
\end{equation*}
$$

giving

$$
\begin{equation*}
\Gamma(f)=\frac{\delta(f)}{2}+\frac{1}{\pi i} \sum_{\ell: \text { odd }} \frac{1}{\ell} \delta\left(f-\frac{2^{n-m-1} \ell}{w \Delta}\right) \tag{25}
\end{equation*}
$$

Function $\hat{\gamma}(t)$ is a time-shifted version of $\gamma(t)$ and so

$$
\hat{\Gamma}(f)=e^{-\frac{\pi i \Delta f}{2^{n-m}}} \Gamma(f)
$$



Fig. 7. Generation of $v(t-\Delta)$ using a rising-edge-triggered D-FF sampling the time-shifted ideal periodic squarewave $\hat{\gamma}(t)$.

$$
\begin{equation*}
\hat{\Gamma}(f)=\frac{\delta(f)}{2}+\frac{1}{\pi i} \sum_{\ell: \text { odd }} \frac{e^{-\frac{\pi i \ell}{2 w}}}{\ell} \delta\left(f-\frac{2^{n-m-1} \ell}{w \Delta}\right) \tag{26}
\end{equation*}
$$

The Fourier transform of $\theta(t)$ is given [30] by

$$
\begin{equation*}
\Theta(f)=\frac{1}{\Delta} \sum_{r=-\infty}^{\infty} \delta\left(f-\frac{r}{\Delta}\right) \tag{27}
\end{equation*}
$$

The convolution $\hat{\Gamma} * \Theta$ is derived using the identity $\delta(f-$ $a) * \delta(f-b)=\delta(f-a-b)$, along with (26) and (27). The result is (28), see next page. Finally, the Fourier transform of the pulse $p(t)$ is ${ }^{8}$

$$
\begin{equation*}
P(f)=\int_{-\Delta}^{0} e^{-2 \pi i f t} d t=\Delta e^{\pi i f \Delta} \operatorname{sinc}(f \Delta) \tag{29}
\end{equation*}
$$

By replacing (28) and (29) into (23), doing some algebraic manipulation, and using the fact that $\operatorname{sinc}(0)=1$ and $\operatorname{sinc}(q)=0$ for all $q \in \mathbb{Z}-\{0\}$ we get the spectrum of the FA in (30), see next page. Finally, by applying the inverse Fourier transform,

[^5]\[

$$
\begin{equation*}
(\hat{\Gamma} * \Theta)(f)=\frac{1}{2 \Delta} \sum_{r} \delta\left(f-\frac{r}{\Delta}\right)+\frac{1}{\pi \Delta i} \sum_{\substack{r, \ell \\ \ell: \text { odd }}} \frac{e^{-\frac{\pi i \ell}{2 w}}}{\ell} \delta\left(f-\frac{r w+2^{n-m-1} \ell}{w \Delta}\right) \tag{28}
\end{equation*}
$$

\]

$$
\begin{equation*}
V(f)=\frac{\delta(f)}{2}+\frac{1}{\pi i} \sum_{\substack{r, \ell \\ \ell: \text { odd }}}(-1)^{r} \cdot \frac{e^{\frac{\pi i \ell}{2 w}\left(2^{n-m}-1\right)}}{\ell} \cdot \operatorname{sinc}\left(\frac{r w+2^{n-m-1} \ell}{w}\right) \cdot \delta\left(f-\frac{r w+2^{n-m-1} \ell}{w \Delta}\right) \tag{30}
\end{equation*}
$$

$$
\begin{equation*}
v(t)=\frac{1}{2}+\frac{1}{\pi} \sum_{\substack{r, \ell \\ \ell: \text { odd }}} \frac{(-1)^{r}}{\ell} \cdot \operatorname{sinc}\left(\frac{r w+2^{n-m-1} \ell}{w}\right) \cdot \sin \left(\frac{\left(2^{n-m}-1\right) \ell \pi}{2 w}+\frac{2\left(r w+2^{n-m-1} \ell\right) \pi}{w \Delta} t\right) \tag{32}
\end{equation*}
$$

$$
\begin{equation*}
v(t)=\int_{-\infty}^{\infty} V(f) e^{2 \pi i f t} d f \tag{31}
\end{equation*}
$$

to (30), we derive (32), see above, for the FA's output signal in the time domain.

## A. Frequency Components of FA's Spectrum

It is worth identifying the frequency components in FA's spectrum and deriving the amplitude of some of them. The frequencies of the terms in (31) and (32) are

$$
\begin{equation*}
f_{r, \ell}=\frac{r w+2^{n-m-1} \ell}{w \Delta} \tag{33}
\end{equation*}
$$

where $\ell$ is odd. Therefore, the set of values of $f_{r, \ell}$ is

$$
\begin{equation*}
\mathcal{F}=\left\{f_{r, \ell} \mid r, \ell \in \mathbb{Z}, \ell: \text { odd }\right\} \tag{34}
\end{equation*}
$$

Note that $f_{r, \ell}$ and $-f_{r, \ell}$ correspond to the same frequency component in (31) and (32). We set $\ell=2 p+1$ and let $p, r$ $\in \mathbb{Z}$ take every integer value. We also define

$$
\begin{equation*}
g=\operatorname{gcd}\left(w, 2^{n-m}\right) \tag{35}
\end{equation*}
$$

and use it to express $f_{r, \ell}$ as

$$
\begin{equation*}
f_{r, \ell}=f_{\mathrm{av}}\left(1+\frac{g}{2^{n-m-1}}\left(r \frac{w}{g}+p \frac{2^{n-m}}{g}\right)\right) \tag{36}
\end{equation*}
$$

where we used $\ell=2 p+1$ and $f_{\text {av }}$ is defined in (9).
Because of (35), integers $w / g$ and $2^{n-m} / g$ are coprime and because $r$ and $p$ can take any integer values, we conclude that

$$
r \frac{w}{g}+p \frac{2^{n-m}}{g}
$$

takes all integer values as well [31]. Therefore, from (36) the set of values of $f_{r, \ell}$ is

$$
\begin{equation*}
\mathcal{F}=\left\{\left.f_{\mathrm{av}}\left(1+j \frac{g}{2^{n-m-1}}\right) \right\rvert\, j \in \mathbb{Z}\right\} . \tag{37}
\end{equation*}
$$

Note that some of these frequency components may not exist in the spectrum because their corresponding terms in (32) cancel each other.

It is convenient to define the parameter

$$
\begin{equation*}
L=\frac{2^{n-m}}{g} \tag{38}
\end{equation*}
$$

and use it to classify the set of frequencies (37). By the definition of $g, L \in\left\{1,2, \ldots, 2^{n-m}\right\}$. We have three possible cases:
$L=1$, then

$$
\begin{equation*}
\mathcal{F}=\left\{(1+2 j) f_{\mathrm{av}} \mid j \in \mathbb{Z}\right\} . \tag{39}
\end{equation*}
$$

$L=2$, then

$$
\begin{equation*}
\mathcal{F}=\left\{j f_{\mathrm{av}} \mid j \in \mathbb{Z}\right\} \tag{40}
\end{equation*}
$$

$L>2$, then $g / 2^{n-m-1}=1 / 2^{h^{\prime}}, h^{\prime} \geq 1$, and

$$
\begin{equation*}
\mathcal{F}=\left\{\left.j \frac{g}{2^{n-m-1}} f_{\mathrm{av}} \right\rvert\, j \in \mathbb{Z}\right\} . \tag{41}
\end{equation*}
$$

Note that 1) and 2) can also be concluded from the rising and falling times of the pulses in $v(t),(7)$ and (8), respectively. Specifically $L=1$ implies a perfect $50 \%$ duty-cycle periodic output, whereas $L=2$ implies a perfect periodic output with duty-cycle different from $50 \%$.

Note also that a value in $\mathcal{F}$ may be attained by $f_{r, \ell}$ for more than one pair $(r, \ell)$. Moreover, there is no distinction between $f_{r, \ell}$ and $-f_{r, \ell}$ in (32). To calculate the amplitude of a frequency component, say of frequency $f_{r_{0}, \ell_{0}}, \ell_{0}$ : odd, one can use Lemma 2 in the Appendix, which states that given the pair $\left(r_{0}, \ell_{0}\right)$, it is $\left|f_{r_{0}, \ell_{0}}\right|=\left|f_{r, \ell}\right|$ if and only if ${ }^{9}$ $(r, \ell) \in \mathcal{A} \cup \mathcal{B}$ where the two sets are

$$
\begin{equation*}
\mathcal{A}=\left\{\left.\left(r_{0}-j \frac{2^{n-m}}{g}, \ell_{0}+j \frac{2 w}{g}\right) \right\rvert\, j \in \mathbb{Z}\right\} \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{B}=\left\{\left.\left(-r_{0}-j \frac{2^{n-m}}{g},-\ell_{0}+j \frac{2 w}{g}\right) \right\rvert\, j \in \mathbb{Z}\right\} \tag{43}
\end{equation*}
$$

Therefore, we can decompose the output $v(t)$ into the sum of its (non-negative)-frequency components $v_{f}(t)$,

$$
\begin{equation*}
v(t)=\frac{1}{2}+\sum_{f \in\left\{\left|f^{\prime}\right| \mid f^{\prime} \in \mathcal{F}\right\}} v_{f}(t), \tag{44}
\end{equation*}
$$

where component ${ }^{10} v_{f}(t)$ of frequency $f$ is given by

$$
\begin{equation*}
v_{f}(t)=\frac{\operatorname{sinc}(\Delta f)}{\pi} \sum_{(r, \ell) \in \mathcal{A} \cup \mathcal{B}} \frac{(-1)^{r}}{\ell} \sin \left(\frac{\left(2^{n-m}-1\right) \ell \pi}{2 w}+2 \pi f_{r, \ell} t\right) \tag{45}
\end{equation*}
$$

and $\mathcal{A}$ and $\mathcal{B}$ are defined by (42) and (43), where $\left(r_{0}, \ell_{0}\right)$, $\ell_{0}$ : odd is a (any) pair satisfying $f_{r_{0}, \ell_{0}}=f$.

## B. Exact Amplitude and Phase of the Average-Frequency Component

In several applications it is important to know the power of the average-frequency component, $v_{f_{\mathrm{av}}}(t)$ [12], [29]. From (9) and (33), we have that $f_{\mathrm{av}}=f_{0,1}$ and so for $r_{0}=$ 0 and $\ell_{0}=1$, sets $\mathcal{A}$ and $\mathcal{B}$ become

$$
\begin{equation*}
\mathcal{A}_{0,1}=\left\{\left.\left(-j \frac{2^{n-m}}{g}, 1+j \frac{2 w}{g}\right) \right\rvert\, j \in \mathbb{Z}\right\} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{B}_{0,1}=\left\{\left.\left(-j \frac{2^{n-m}}{g},-1+j \frac{2 w}{g}\right) \right\rvert\, j \in \mathbb{Z}\right\} \tag{47}
\end{equation*}
$$

By definition $g \leq w$. If $g=w$, then our assumption $w \geq$ $2^{n-m}$ leads to $L=1$ which implies that $v(t)$ is the ideal $50 \%$ duty-cycle squarewave, see comments in Section IV-A and [16].




Fig. 8. The amplitude, $\left\|v_{f_{\text {av }}}\right\|$, of the average frequency, $f_{\text {av }}$, component for $n=7, m=2,4,6$, and $w=2^{n-m}, \ldots, 2^{n}$.

From here on, we assume that $L \geq 2$, and so $g<w$, which implies that integer ${ }^{11} w / g \geq 2$. In this case, there exists no pair of integers $j, j^{\prime}$ satisfying the equation

$$
1+j \frac{2 w}{g}=-1+j^{\prime} \frac{2 w}{g}
$$

and so sets $\mathcal{A}_{0,1}$ and $\mathcal{B}_{0,1}$ are disjoint. Then, (45) gives (51), which provides (52) via (38), (46), and (47).

Identities $\sin (k \pi+a)=(-1)^{k} \sin (a), k \in \mathbb{Z}$, and $\sin (\alpha$ $+\beta)=\sin (\alpha) \cos (\beta)+\sin (\beta) \cos (\alpha)$, and the corresponding ones for the cosine, as well as setting

$$
\psi(t)=\frac{2^{n-m}-1}{2 w} \pi+2 \pi f_{\mathrm{av}} t
$$

transform (52) into (53), where the sine and cosine terms have been grouped respectively. Using (61) in Fact 1 in the Appendix with parameters $a=g /(2 w)$ and $x=\pi / g$, and noting that $(\pi / g) \bmod (2 \pi)=\pi / g$, the first sum on the right side of (53) is expressed as

$$
\begin{equation*}
\sum_{j \in \mathbb{Z}} \frac{\frac{4 w}{g} j}{\left(\frac{2 w}{g} j\right)^{2}-1} \sin \left(\frac{\pi}{g} j\right)=\frac{g \pi}{w} \cdot \frac{\sin \left(\frac{(g-1) \pi}{2 w}\right)}{\sin \left(\frac{g \pi}{2 w}\right)} \tag{48}
\end{equation*}
$$

Similarly, using (62) in Fact 1 in the Appendix with the same parameter values, the second sum on the right side of (53) is expressed as

[^6]\[

$$
\begin{align*}
& \frac{\pi}{\operatorname{sinc}\left(\Delta f_{\mathrm{av}}\right)} \cdot v_{f_{\mathrm{av}}}=\sum_{(r, \ell) \in \mathcal{A}_{0,1} \cup \mathcal{B}_{0,1}} \frac{(-1)^{r}}{\ell} \sin \left(\frac{\left(2^{n-m}-1\right) \ell \pi}{2 w}+\frac{2\left(r w+2^{n-m-1} \ell\right) \pi}{w \Delta} t\right)  \tag{51}\\
& \frac{\pi}{\operatorname{sinc}\left(\Delta f_{\mathrm{av}}\right)} \cdot v_{f_{\mathrm{av}}}=\sum_{j \in \mathbb{Z}} \frac{(-1)^{L j}}{1+\frac{2 w}{g} j} \cdot \sin \left(\frac{2^{n-m}-1}{g} j \pi+\frac{2^{n-m}-1}{2 w} \pi+2 \pi f_{\mathrm{av}} t\right) \\
& +\sum_{j \in \mathbb{Z}} \frac{(-1)^{L j}}{-1+\frac{2 w}{g} j} \cdot \sin \left(\frac{2^{n-m}-1}{g} j \pi-\frac{2^{n-m}-1}{2 w} \pi-2 \pi f_{\mathrm{av}} t\right)  \tag{52}\\
& \frac{\pi}{\operatorname{sinc}\left(\Delta f_{\mathrm{av}}\right)} \cdot v_{f_{\mathrm{av}}}=\cos (\psi(t)) \cdot \sum_{j \in \mathbb{Z}} \frac{\frac{4 w}{g} j}{1-\left(\frac{2 w}{g} j\right)^{2}} \cdot \sin \left(\frac{\pi}{g} j\right)+\sin (\psi(t)) \cdot \sum_{j \in \mathbb{Z}} \frac{2}{1-\left(\frac{2 w}{g} j\right)^{2}} \cdot \cos \left(\frac{\pi}{g} j\right)  \tag{53}\\
& v_{f_{\mathrm{av}}}(t)=\frac{2}{\pi L} \cdot \frac{\sin \left(\frac{2^{n-m} \pi}{2 w}\right)}{\sin \left(\frac{g \pi}{2 w}\right)} \cdot \sin \left(2 \pi f_{\mathrm{av}}\left(t+\frac{L-1}{2 L} \Delta\right)\right) \tag{54}
\end{align*}
$$
\]

$$
\begin{equation*}
\sum_{j \in \mathbb{Z}} \frac{2}{1-\left(\frac{2 w}{g} j\right)^{2}} \cos \left(\frac{\pi}{g} j\right)=\frac{g \pi}{w} \cdot \frac{\cos \left(\frac{(g-1) \pi}{2 w}\right)}{\sin \left(\frac{g \pi}{2 w}\right)} \tag{49}
\end{equation*}
$$

After substituting (48) and (49) into (53) and performing some algebraic manipulation, we conclude that the average frequency component of the output signal $v(t)$ is given by (54), which is valid for $w \geq 2^{n-m}$ and $L \geq 2$. If $L=1$, then $v(t)$ is $0-1$ (digital) $50 \%$ duty-cycle periodic squarewave [16], and so $f_{\text {av }}$ is also the fundamental frequency with corresponding amplitude ${ }^{12}\left\|v_{f_{\mathrm{av}}}\right\|=2 / \pi$.

Combining the previous conclusions, we have for $w \geq$ $2^{n-m}$ and any value of $L$

$$
\begin{equation*}
\left\|v_{f_{\mathrm{av}}}\right\|=\frac{2}{\pi L} \cdot \frac{\sin \left(\frac{2^{n-m} \pi}{2 w}\right)}{\sin \left(\frac{g \pi}{2 w}\right)} \tag{50}
\end{equation*}
$$

Fig. 8 shows $\left\|v_{f_{\text {av }}}\right\|$ for $n=7, m=2,4,6$, and $w=$ $2^{n-m}, \ldots, 2^{n}$. Note that for large values of $w,\left\|v_{f_{\mathrm{av}}}\right\|$ approaches $2 / \pi$, which is the amplitude of the fundamental frequency component of a $0-1$ (digital) $50 \%$ duty-cycle periodic squarewave.

## V. Measurements and Simulation

The FA has been implemented in a Xilinx field-programmable gate array (FPGA) (Xilinx Inc., San Jose, CA) and its spectrum has been measured to verify the

[^7]

Fig. 9. Measured and analytically derived (dots) spectrum of the flying adder FPGA implementation with $n=8, m=4$, and $w=142$.
developed theory. The implementation was with $n=8$ and $m=4$. The $2^{m}=16$ clock phases were generated using shift registers clocked at 50 MHz and resulting in clock frequency of the FA, $f_{\text {clk }}=(50 / 16) \mathrm{MHz}=3.125 \mathrm{MHz}$.

Fig. 9 shows the measured spectrum of the implemented FA when $w=142$, along with the spectrum derived using (30), indicated by the dots. The strongest frequency component is at

$$
f_{\mathrm{av}}=\frac{2^{n-m-1}}{w \Delta}=\frac{2^{n-1} f_{\mathrm{clk}}}{w} \approx 2.817 \mathrm{MHz}
$$

and because $w=142 \geq 2^{n-m}=16, g=\operatorname{gcd}\left(2^{n-m}, w\right)=2$, and $L=8>2$, the expected frequencies of the predicted spectrum are given by (41).

The matching between the measurements and theory is very good for all predicted frequency components within the measured frequency span ${ }^{13}$ with relative errors less

[^8]than 1 dB . There are, however, some additional components, at least 50 dB below $f_{\text {av }}$ which are not predicted by the theory. Most probably, they are due to leakage of the spectrum of internal states $x_{k}, y_{k}$ to the output, or are the impact of the lack of a perfect $50 \%$ duty cycle of the driving clock [16].

Both frequency and time-domain expressions of the FA's output perfectly match the results of MATLAB simulation.

## VI. Conclusions

This work has rigorously derived analytic expressions of the exact spectrum and output signal of the flyingadder frequency synthesizer and compact closed-form expressions of the amplitude and phase of flying-adder's average-frequency component, and has expressed the fly-ing-adder's output as a uniformly sampled-and-held ideal $50 \%$ duty-cycle squarewave.

## Appendix

Lemma 1: If positive integers $m, n$, and $w$ are such that $m \leq n$ and $2^{n-m} \leq w<2^{n}$, then the definitions of $\delta_{k}, t_{j}$, and $\tau_{j}$, given by (4), (5), and (6) respectively, imply (7) and (8).

Proof: Using inequality $[a-b] \leq[a]-[b] \leq\lceil a-b\rceil$, which is valid for all $a, b \in R,(\lceil \rceil$ is the ceiling function $)$ we get

$$
\left[\frac{w}{2^{n-m}}\right] \leq\left[\frac{k w}{2^{n-m}}\right]-\left[\frac{(k-1) w}{2^{n-m}}\right] \leq\left\lceil\frac{w}{2^{n-m}}\right\rceil
$$

which with our assumption of $2^{n-m} \leq w<2^{n}$ implies

$$
\begin{equation*}
1 \leq\left[\frac{k w}{2^{n-m}}\right]-\left[\frac{(k-1) w}{2^{n-m}}\right] \leq 2^{m} \tag{55}
\end{equation*}
$$

Set

$$
z=\left[\frac{k w}{2^{n-m}}\right]-\left[\frac{(k-1) w}{2^{n-m}}\right]
$$

and consider the case $z=2^{m}$ first. It is implied by (3) that $y_{k}=y_{k-1}$, and so definition (4) gives

$$
\begin{equation*}
\delta_{k}=2^{m}=\left[\frac{k w}{2^{n-m}}\right]-\left[\frac{(k-1) w}{2^{n-m}}\right] \tag{56}
\end{equation*}
$$

Next, suppose that $z<2^{m}$, which, along with (3) and (55), implies that $y_{k} \neq y_{k-1}$ as well as $z \bmod 2^{m}=z$. Identity

$$
(x \bmod a \pm y \bmod a) \bmod a=(x \pm y) \bmod a
$$

is valid for all integers $x, y, a$ with $a \geq 1$ and implies

$$
\left(y_{k}-y_{k-1}\right) \bmod 2^{m}=z \bmod 2^{m}=z
$$

Again, from definition (4) we have $\delta_{k}=z$, which, along with the previous case resulting in (56), implies that

$$
\begin{equation*}
\delta_{k}=\left[\frac{k w}{2^{n-m}}\right]-\left[\frac{(k-1) w}{2^{n-m}}\right] \tag{57}
\end{equation*}
$$

for all values of $w$ such that $2^{n-m} \leq w<2^{n}$. Eq. (7) and (8) result directly from replacing (57) into (5) and (6), respectively.

Lemma 2: Let $r_{0}, \ell_{0} \in \mathbb{Z}, \ell_{0}$ : odd. Then, $\left|f_{r_{0}, \ell_{0}}\right|=\left|f_{r, \ell}\right|$ if and only if $(r, \ell) \in \mathcal{A} \cup \mathcal{B}$, where sets $\mathcal{A}$ and $\mathcal{B}$ are defined by (42) and (43), respectively.

Proof: To prove the lemma it is sufficient to show that

$$
f_{r, \ell}=f_{r_{0}, \ell_{0}} \Leftrightarrow\left\{\begin{array}{l}
r=r_{0}-j \frac{2^{n-m}}{g}  \tag{58}\\
\ell=\ell_{0}+j \frac{2 w}{g}
\end{array}\right.
$$

for some $j \in \mathbb{Z}$, and

$$
f_{r, \ell}=-f_{r_{0}, \ell_{0}} \Leftrightarrow\left\{\begin{array}{l}
r=-r_{0}-j^{\prime} \frac{2^{n-m}}{g}  \tag{59}\\
\ell=-\ell_{0}+j^{\prime} \frac{2 w}{g}
\end{array}\right.
$$

for some $j^{\prime} \in \mathbb{Z}$. Eq. (36) implies that $f_{r_{0}, \ell_{0}}=f_{r, \ell}$ if and only if

$$
\begin{equation*}
\left(r-r_{0}\right) \frac{w}{g}+\left(p-p_{0}\right) \frac{2^{n-m}}{g}=0 \tag{60}
\end{equation*}
$$

where $\ell=2 p+1$ and $\ell_{0}=2 p_{0}+1$. Because $w / g$ and $2^{n-m} / g$ are coprime integers, (60) holds if and only if [31] $r=r_{0}-j 2^{n-m} / g$ and $p=p_{0}+j w / g$ for some $j \in \mathbb{Z}$, or equivalently if and only if $r=r_{0}-j 2^{n-m} / g$ and $\ell=\ell_{0}+$ $2 j w / g$ for some $j \in \mathbb{Z}$. (Note that $\ell$ is odd because $\ell_{0}$ is odd and $w / g$ is an integer.) This proves (58). Equivalence (59) is established similarly.

Fact 1: For every real number $a, x$ with $a \notin \mathbb{Z}[32]$,

$$
\begin{gather*}
\sum_{k=1}^{\infty} \frac{k \sin (k x)}{k^{2}-a^{2}}=\frac{\pi \sin (a \pi-a(x \bmod 2 \pi))}{2 \sin (a \pi)}  \tag{61}\\
\sum_{k=1}^{\infty} \frac{\cos (k x)}{k^{2}-a^{2}}=\frac{1}{2 a^{2}}-\frac{\pi \cos (a \pi-a(x \bmod 2 \pi))}{2 a \sin (a \pi)}, \tag{62}
\end{gather*}
$$

where for $x, y \in \mathbb{R}, y>0, x \bmod y$ is defined as $x-[x / y] y$.

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[^1]:    ${ }^{1}$ The identity is valid for all non-negative integers $a, c, d$, with $c \geq d$.
    ${ }^{2}$ Spikes are considered as zero-length pulses, i.e., with rising and falling edges appearing at the same time.

[^2]:    ${ }^{3}$ That is, $x_{k(t)}, y_{k(t)}$, etc.

[^3]:    ${ }^{4} \mathbb{Z}$ is the set of integer numbers.
    ${ }^{5}$ We consider the time intervals $[r w \Delta,(r+1) w \Delta)$.
    ${ }^{6}$ The derivation of $f_{\text {av }}$ here is valid for $w \geq 2^{n-m}$ and it is based on the $w \Delta$-seconds-long period of $v(t)$. A more general and stronger result is established in [15], where $f_{\mathrm{av}}$ is derived for all values of $w$ based on the fundamental period of $v(t)$.

[^4]:    ${ }^{7}$ This way, we avoid sampling discontinuities at the transitions of the pulses that would create mathematical complications.

[^5]:    ${ }^{8}$ The definition of the sinc function used here is $\operatorname{sinc}(x)=\sin (\pi x) /(\pi x)$ for $x \neq 0$ and 1 otherwise.

[^6]:    ${ }^{11}$ Recall the definition of $g$ in (35).

[^7]:    $12\left\|v_{f_{\mathrm{av}}}\right\|$ denotes the amplitude of $v_{f_{\mathrm{av}}}$.

[^8]:    ${ }^{13}$ The spectrum analyzer used operates at above 9 kHz . This explains the larger error in the amplitude of the dc component.

