Theory of Flying-Adder Frequency Synthesizers—Part I: Modeling, Signals' Periods and Output Average Frequency

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***Abstract*—This is a rigorous mathematical theory of the operation of the flying-adder (FA) frequency synthesizer (also called direct digital period synthesizer). The paper consists of two parts: Part I presents a detailed mathematical model of the FA synthesizer, capturing the relationships between the properties of the FA's output and internal signals and the FA's parameters. The counting of the rising edges in the FA's multiplexer's output establishes a discrete-time index that is used to analytically derive the fundamental discrete-time periods of all FA's signals. The continuous-time intervals between the rising edges are calculated and used to derive the fundamental continuous-time periods of the signals from the corresponding discrete-time ones. It is shown that the FA behaves differently within different ranges of the frequency word, and the practically useful range is identified. The FA's output average frequency, along with its maximum and minimum values, is analytically derived by calculating the number of cycles in the output signal within a fundamental continuous-time period of it. The relationship between the average and the fundamental output frequencies is also established, indicating the potential frequencies and density of output spurious frequency components. Part II of the paper characterizes the timing structure of the output signal, providing analytical expressions of the pulses' locations, analytical strict bounds of the timing irregularities, and exact analytical expressions of several standard jitter metrics. Spectral properties of the output waveform are presented, including the dominance of the frequency component at the average frequency, and analytical expressions of the dc value and average power of the output signal are derived. The FA has been implemented in a Xilinx Spartan-3E field-programmable gate array, and spectral measurements are presented, confirming the theoretical results. Extensive MATLAB simulation has also been used to generate numerous examples illustrating the developed theory.**

***Index Terms*—Clock generation, digital-to-frequency converter (DFC), direct digital period synthesis, direct digital synthesis (DDS), flying adder (FA), frequency synthesis, jitter, phase accumulator, phase synthesis, spurs, truncation.**

I. INTRODUCTION

FREQUENCY synthesis is a critical part in a very wide range of modern electronic systems. A large number of frequency synthesis architectures are available in the literature, addressing the often contradictory requirements of the broad spectrum of applications [1]–[6].

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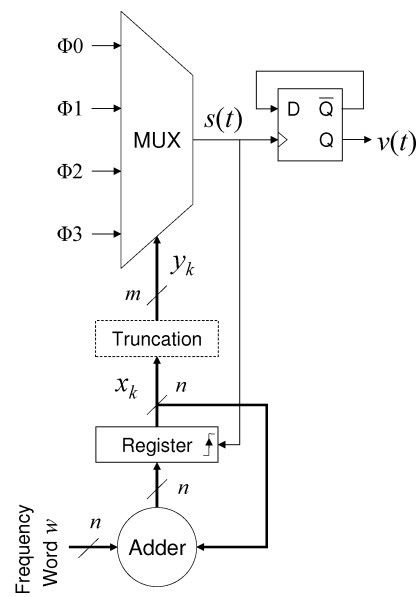

Fig. 1. FA (shown here with $m = 2$, i.e., 2^m input phases).

The flying-adder (FA) frequency synthesizer [7], [8], which is also called direct digital period synthesizer (DDPS) [9], or a digital-to-frequency converter (DFC) [10], is an architecture for generating almost-periodic digital signals of a desired average frequency [11], based on a frequency reference clock. The FA is shown in abstract form in Fig. 1, driven by a family of uniformly phase-shifted copies of a periodic square wave that can be generated by a ring oscillator, for example, as shown in Fig. 2.

Structures similar to FA, involving a phase-switching prescaler and a digital phase accumulator, have been used in the past [1]–[3], [12]–[18]. However, the FA appeared as an independent frequency synthesis circuit first[1] in [7] and was a patented invention in [8]. A long list of related publications followed, including [9]–[11] and [19]–[33].

The FA architecture results in simple compact fully digital implementations offering design convenience and good period resolution. These properties have made FA a successful circuit block in several commercial integrated circuit products [25].

Although the FA has the aforementioned advantages (at least as a digital period synthesizer) it also suffers from highly spurious output content (for most of the frequency words) due to phase truncation. Work has been done to estimate the output

[1]To the best of the author's knowledge.

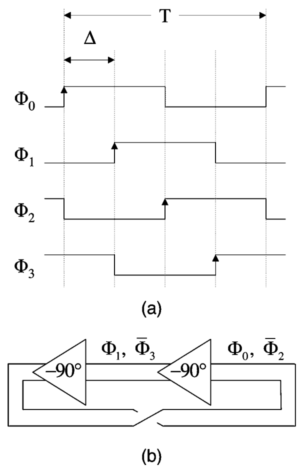

Fig. 2. 2^m input phases to the FA, typically generated by a ring oscillator. It is $f_{\rm clk} = 1/T = 1/(2^m\Delta)$ (shown here with $m = 2$).

spurious content [21], [23], [26]. In some sense, the mechanism generating the spurs in the FA is similar to that generating the spurs in direct digital synthesis (DDS), which has extensively been studied in [12], [14], [15], [34]–[44], and other publications.

There is, however, a significant structural difference between the abstract DDS and FA architectures. In DDS, the phase accumulator is updated at the (fixed) reference clock rate in contrast to FA, which has a feedback path updating its register at a variable rate (for most values of the frequency word).

This work is a rigorous mathematical theory of operation of the FA frequency synthesizers. It responds to the request in [10] and [11] for a rigorous analysis of the FA. The paper consists of two parts [45]. The present paper (Part I) presents a detailed mathematical model of the FA, capturing the relationships between its signals and parameters.

The *fundamental discrete-time periods* of the FA's signals are mathematically derived using the rising edges of the multiplexer's (MUX's) output signal as our discrete-time reference (see Fig. 1). The fundamental discrete-time periods are converted into their corresponding *fundamental continuous-time periods* by analytically deriving the time intervals between consecutive rising edges of the MUX's output signal $s(t)$. Knowing the fundamental periods, and not only any periods (multiples of the fundamental ones), is important because the spectra of the signals are composed of harmonics of the fundamental frequencies.

Note that the spectra of internal signals are also of concern because they can leak to the output of the synthesizer through direct parasitic coupling or through the power supply network. Such signal leakage is shown in the spectrum of the field-programmable gate array FA implementation presented in Part II.

Using the FA's output fundamental continuous-time period and by counting the output pulses within it, the output *average frequency* is analytically derived. Conditions are presented under which the output average frequency is monotonically related to the frequency word.

In Part I, Section II illustrates the behavior of the FA, captures it in equations, and introduces the notation used throughout this two-part paper. Section III derives the discrete-time periods of the signals in the FA. Section IV converts the discrete-time periods of the FA signals into their corresponding continuous-time (real-time) periods. The average frequency of the FA and its harmonic index are analytically derived in Section V.

Part II of the paper is devoted to the timing and spectral properties of the FA's output signal.

II. Operation, Modeling, and Notation of the FA

Consider the FA[2] in Fig. 1 that follows the architectures introduced in [7] and discussed in [11].

A family of 2^m periodic, square-wave, 50% duty-cycle signals[3] of the same frequency $f_{\rm clk}$ and relative phase offsets that form an arithmetic progression of step $-2\pi/2^m$ rad, as shown in Fig. 2(a) for $m = 2$, are fed into the 2^m-to-1 MUX in Fig. 1.

The period of the 2^m clock signals is $T = 1/f_{\rm clk}$, and the phase step corresponds to time delay that is equal to $\Delta = T/2^m$, as shown in Fig. 2. The phase ensemble can be generated by a ring oscillator, as shown in Fig. 2(b).

The MUX is controlled by the m-bit word y_k, which selects the one of the 2^m input phases that propagates to the output of the MUX forming signal $s(t)$.

The nonnegative integer variable k counts the rising edges of $s(t)$. k is our discrete-time reference; specifically, for every $\ell = 1, 2, \ldots$, the discrete-time (period) $k = \ell$ corresponds to the (real) continuous-time interval between the ℓth and $\ell + 1$ rising edges of $s(t)$. As shown in Section IV-A, k does *not* correspond linearly to (real) continuous-time t.

The rising edges of $s(t)$ trigger the n-bit *register*, as shown in Fig. 1, updating its value from x_k to $x_{k+1} = (x_k + w) \bmod 2^n$.

The register's value x_k is truncated by keeping the m most significant bits (MSB). This defines the variable y_k, which controls the MUX and can be expressed as $y_k = x_k \operatorname{div} 2^{n-m}$.

The D-flip-flop (D-FF) is essentially a (frequency) divider by 2, i.e., it produces an output clock cycle for every two consecutive rising edges (or spikes—considered as pairs of rising and falling edges) of $s(t)$. Without loss of generality, we assume that the initial value of the register, at $k = 0$, is $x_0 = 0$ and that of the D-FF is $Q = 0$.

The following three examples illustrate major points of the operation of FA, which are summarized in the corresponding remarks at the end of the examples.

Example 2.1: The operation of the FA in Fig. 1 is illustrated in Fig. 3(a)–(g) for the case of $n = 4$, $m = 2$, and frequency word $w = 7$. Fig. 3(a)–(g) is the result of MATLAB simulation of the FA structure in Fig. 1.

The x axis is the (real) continuous time in multiples of Δ. The four input phases ϕ_r, $r = 0, 1, 2, 3$, are shown in Fig. 3(a). The selection of the input phase ϕ_r that is propagated to the output, i.e., $s(t) = \phi_r(t)$, is done by the MUX according the value of y_k. The time intervals that each of the phases ϕ_r, $r = 0, 1, 2, 3$, is selected are indicated with thick line segments in Fig. 3(a).

Fig. 3(b)–(d) shows the continuous-time waveforms[4] of the discrete-time sequences $\{x_k\}$, $\{y_k\}$, and $\{d_k\}$,

[2]For presentation purposes, 2^m, $m = 2$, input phases are used.

[3]$\phi_r(t) = 1/2 + \operatorname{sgn}(\sin(2\pi f_{\rm clk}t - 2\pi r/2^m))/2, r = 0, 1, \ldots, 2^m - 1$.

[4]The rising edges counter k is a function of continuous-time t, and thus, the sequences $\{x_k\}$, $\{y_k\}$, and $\{d_k\}$ can be considered as the functions of continuous time, i.e., $x_{k(t)}$, $y_{k(t)}$, and $d_{k(t)}$, respectively.

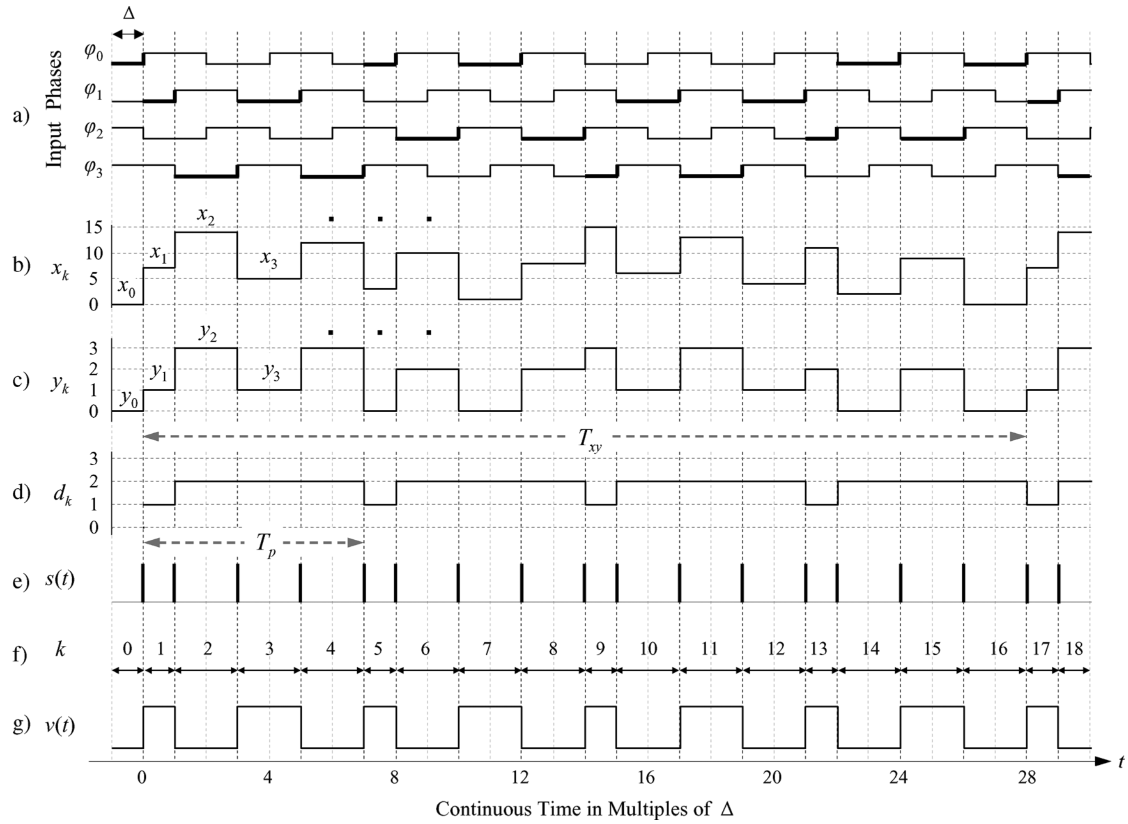

Fig. 3. FA, $n = 4$, $m = 2$, $w = 7$.

where the nonnegative integer sequence $\{d_k\}$ is defined as $d_k = (y_k - y_{k-1}) \bmod 2^m$. It is shown later in the paper that the continuous time between two consecutive rising edges of $s(t)$ is equal to $d_k\Delta$ if $d_k > 0$; otherwise, it is equal to $2^m\Delta$.

The set of parameters $n = 4$, $m = 2$, and $w = 7$ used in this example results in signal $s(t)$, as shown in Fig. 3(e), that is only composed of spikes (in contrast to the cases of Figs. 4 and 5, where $s(t)$ contains pulses, in addition or exclusively).

By its definition, the discrete-time k, as shown in Fig. 3(f), results from counting the rising edges of s. "All action" in the FA takes place at the rising edges of s, and the discrete time has value k between the kth and $k+1$ rising edges of signal s.

The D-FF in the FA counts the spikes (or rising edges) in s modulo 2 (frequency division by 2), resulting in the output signal v shown in Fig. 3(g).

Remark: Note that, for discrete-time k, $d_k > 0$ if and only if the kth rising edge of signal s results in a change of value from y_{k-1} to y_k. Such a change results in a change of the selected phase and a $k+1$ rising edge appearing $d_k\Delta$ seconds later. For example, in Fig. 3(e), the second rising edge (here it is just a spike) of s appears at $t = \Delta$; since $y_1 = 1$ and $y_2 = 3$, we have $d_2 = 2$; indeed, the third rising edge appears at $t = 3\Delta$.

Example 2.2: Fig. 4 captures the operation of the FA when $n = 4$, $m = 2$, and the frequency word is $w = 3$.

As in the previous example, the four input phases ϕ_r, $r = 0$, 1, 2, 3, are shown in Fig. 4(a), and the intervals within which each of them is selected by the MUX are shown in thick line segments.

The important difference w.r.t. the case of Fig. 3 is that, in Fig. 4(c), there are consecutive discrete-time values k for which the value of the sequence $\{y_k\}$ does not change,[5] e.g., for $k = 0, 1$ [see Fig. 4(f)], $y_0 = y_1 = 0$. Similarly, for $k = 4, 5$, $y_4 = y_5 = 3$.

For values of k such that $y_k = y_{k+1}$, both the kth and $k+1$ rising edges are due to the (same) input phase $\phi_{y_k} = \phi_{y_{k+1}}$. Therefore, the $k+1$ rising edge appears $T = 2^m\Delta$ seconds after the kth one.

Observe this case for $k = 4, 5$ in Fig. 4. This explains why the discrete-time periods $k = 1, 5, 9, 13, \ldots$ are $2^m\Delta$ (seconds) long. Here, such (full-cycle) discrete-time intervals are isolated because $2w \geq 2^{n-m}$, which implies that, after two rising edges of $s(t)$, the value of x_k increases by $2w \bmod 2^n$, and thus, y_k changes. Moreover, during every such discrete-time period k, signal s contains a whole cycle of ϕ_{y_k}, as shown in Fig. 4(e), instead of just a spike. Typically, this results in an "irregular" duty cycle of output signal v.

Remark: Since $d_k = (y_k - y_{k-1}) \bmod 2^m$, it is $d_k = 0$ if and only if $y_k - y_{k-1} = 0$, and based on the previous discussion, for every k such that $d_k = 0$, the kth discrete-time interval is $2^m\Delta$ seconds long. Such (full-cycle) intervals may be repeated if $w < 2^{n-m}/2$ because, in some instances, three or more consecutive edges of $s(t)$ are needed to change the value of y_k.

Finally, the time sequences t_ℓ, $\ell = 1, 2, \ldots$, and τ_ℓ, $\ell = 1, 2, \ldots$, as shown between Fig. 4(f) and (g), correspond (by definition) to the rising and falling edges of the output signal v and, therefore, to the even- and odd-number rising edges of s.

Example 2.3: Fig. 5 captures the operation of the FA when $n = 4$, $m = 2$, and the frequency word is $w = 15$.

[5]Although x_k increases modulo 2^n.

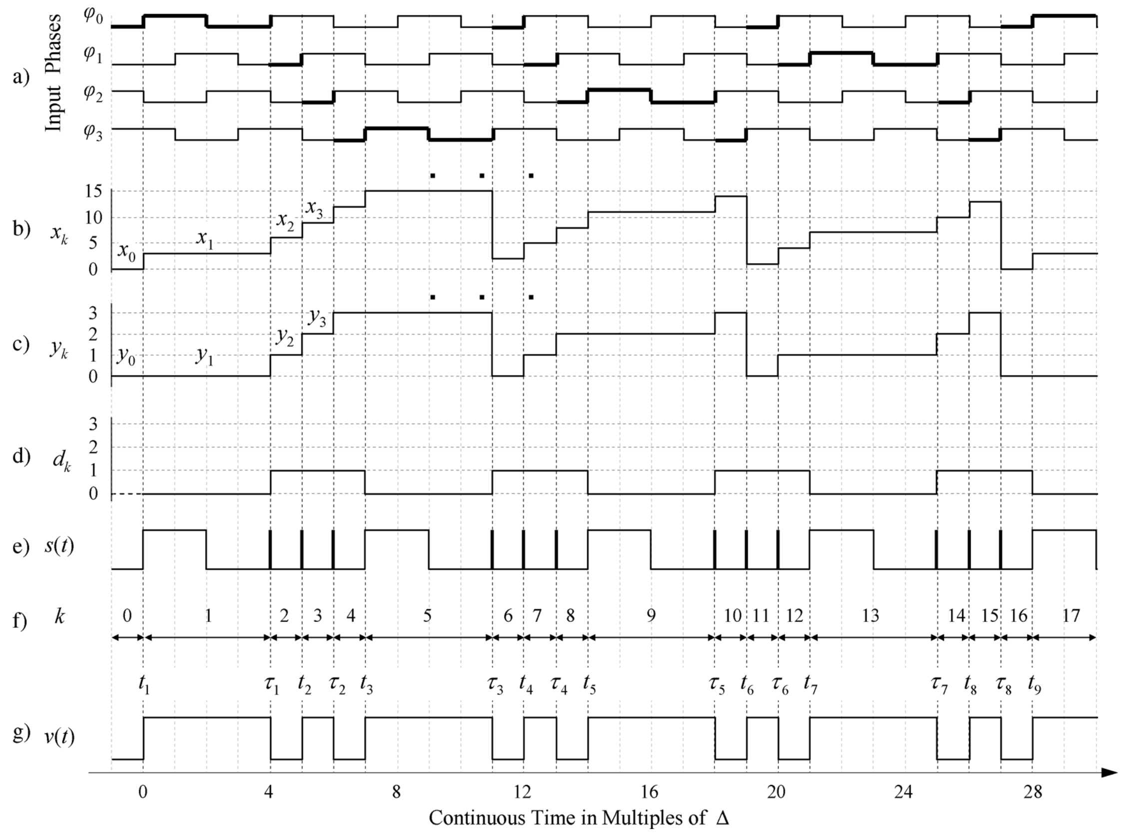

Fig. 4. FA, $n = 4, m = 2, w = 3$.

Again, the four input phases ϕ_r, $r = 0$, 1, 2, 3, are shown in Fig. 5(a), and the intervals within which each of them is selected by the MUX are shown in thick line segments.

Here as well, we see consecutive discrete-time values k for which the sequence $\{y_k\}$ has the same value, e.g., for $k = 1$, 2, 3, 4 [see Fig. 5(f)], it is $y_1 = y_2 = y_3 = y_4 = 3$. (Compare this with Fig. 4 of Example 2.2.)

Remark: Here, we have 3.75 consecutive full-cycle periods T before any change in y_k [see Fig. 5(a)]. Again, observe that zero values of d_k correspond to discrete-time intervals that are full-cycle long, i.e., $2^m\Delta$ seconds.

A. Notation, Definitions, and Basic Relationships

This section lists the definitions of all variables and parameters used throughout this work, along with some of their basic relationships. In addition, the term *fundamental period* is clarified in Section II-A-3.

1) Parameters and Variables: With regard to the following definitions, please refer to Figs. 1–4.

- n: The size of FA's register (bits).
- x_k: The *state* of the FA. It is the n-bit value of the register during the kth discrete-time interval (see below). We refer to the sequence $\{x_k\}$, $k = 0, 1, 2, \ldots$, as the *state sequence*. It is $x_k \in \{0, 1, \ldots, 2^n - 1\}$.
- m: The number of bits controlling the MUX, which selects one of the 2^m input phases. It is $1 \leq m \leq n$.
- y_k: The *truncated state*. It is formed by the m MSB (truncation) of x_k. We refer to the sequence $\{y_k\}$, $k = 0, 1, 2, \ldots$, as the *truncated state sequence*. It is $y_k \in \{0, 1, \ldots, 2^m - 1\}$.
- w: The *frequency word* that is fed to the adder. It is n-bit long, and thus, $w \in \{0, 1, \ldots, 2^n - 1\}$. The case $w = 0$ is trivial and is not considered.
- ϕ_r, $r = 0, 1, \ldots, 2^m - 1$: The *input phases*, which are a family of 2^m periodic, square-wave, 50% duty-cycle (clock) signals of the same frequency $f_{\text{clk}} = 1/T$ and relative phase offsets that form an arithmetic progression with step of $-2\pi/2^m$ rad. The phase step corresponds to time delay that is equal to $\Delta = T/2^m$.
- f_{clk}: The *frequency* of the 2^m input phases.
- T: The *period* of the 2^m input phases, $T = 1/f_{\text{clk}}$.
- Δ: The relative time delay between consecutive input phases ϕ_r, $r = 0, 1, \ldots, 2^m - 1$. It is $\Delta = T/2^m$.
- $s(t)$: The *output signal of the MUX*. It coincides with the input phase $\phi_{y_k}(t)$ during the discrete-time interval k.
- k: The *discrete-time* index $k = 0, 1, 2, \ldots$ counts the rising edges[6] of signal $s(t)$ at the output of the MUX. Index k is our discrete-time reference and can be defined as the function of continuous-time $k(t) =$ number of rising edges in signal s within the continuous-time interval $(0^-, t^+)$.
- kth *discrete-time interval*: The (continuous) time interval between the kth and $k + 1$ rising edges of $s(t)$. It corresponds to the value of discrete-time k.
- d_k: The *difference sequence*, which is defined using the truncated state sequence, $d_k = (y_k - y_{k-1}) \bmod 2^m$. As

[6]Spikes in $s(t)$, as in Figs. 3(e) and 4(e), are considered to be pairs of rising and falling edges.

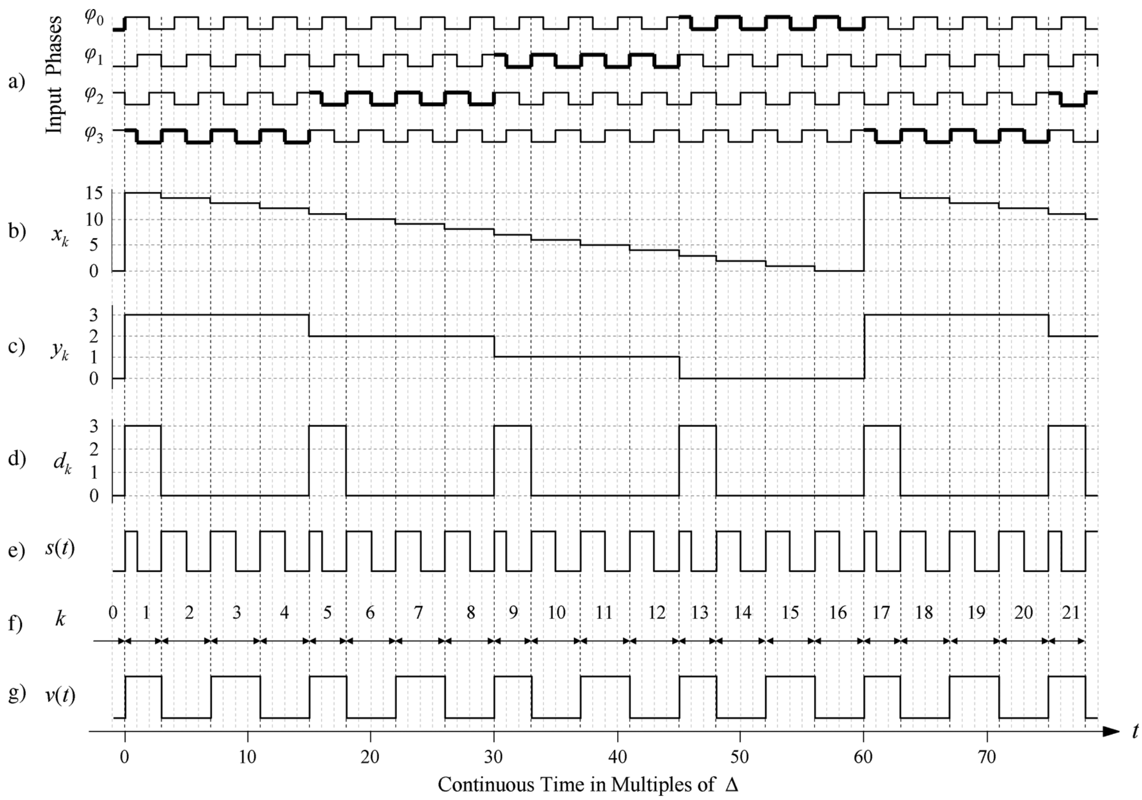

Fig. 5. FA, $n = 4$, $m = 2$, $w = 15$.

mentioned in Example 2.2, if $d_k > 0$, then the continuous-time interval between the kth and $k+1$ rising edges of $s(t)$ is $d_k\Delta$ seconds long.

- δ_k: The *modified difference sequence*, which is defined as
$$\delta_k = \begin{cases} d_k, & \text{if } d_k > 0 \\ 2^m, & \text{otherwise.} \end{cases}$$
The (real) time interval between the kth and $k+1$ rising edges of $s(t)$ is $\delta_k\Delta$ seconds long for every $k = 1, 2, 3, \ldots$ (recall Examples 3–5). This is true for all possible values of parameters n, m, and w.
- $v(t)$: The *output signal* of the FA resulting from counting modulo 2 the rising edges of signal $s(t)$.
- K: The *fundamental*[7] *discrete-time period* of the state and truncated state sequences $\{x_k\}$ and $\{y_k\}$, respectively.
- L: The *fundamental discrete-time period* of the difference and modified difference sequences $\{d_k\}$ and $\{\delta_k\}$, respectively.
- T_{xy}: The *fundamental continuous-time period* of the state and truncated state sequences[8] $\{x_k\}$ and $\{y_k\}$.
- T_p: The *fundamental continuous-time period* of the sequences[9] $\{d_k\}$ and $\{\delta_k\}$ and signal $s(t)$.
- T_v: The *fundamental continuous-time period* of the output signal $v(t)$.
- f_{v_f}: The *fundamental frequency* of the output signal $v(t)$. It is $f_{v_f} = 1/T_v$
- f_{av}: The *average frequency* of the output signal $v(t)$, which is defined as the number of pulses within its fundamental continuous-time period T_v divided by T_v.
- N_{av}: The average frequency is the N_{av} harmonic of the output signal $v(t)$, i.e., $f_{\text{av}} = N_{\text{av}} f_{v_f}$.
- t_ℓ, τ_ℓ: The continuous time at which the ℓth rising/falling edge of $v(t)$ appears, $\ell = 1, 2, 3, \ldots$. The two sequences correspond to the odd and the even rising edges (or spikes) of $s(t)$, respectively.

Remark: All continuous-time intervals and periods we consider start and end at a rising edge (or a spike) of signal $s(t)$. Note that the assumptions and definitions above imply that every rising edge and spike of $s(t)$ appears at continuous-time $t = r\Delta$, for some nonnegative integer r.

2) Basic Relationships: The state of the FA is updated based on $x_{k+1} = (x_k + w) \bmod 2^n$. Without loss of generality, we can assume that $x_0 = 0$, and thus, we have

$$x_k = (kw) \bmod 2^n. \tag{1}$$

Since y_k is derived by keeping the first m (MSB) out of total n bits of x_k

$$y_k = x_k \text{ div } 2^{n-m} \doteq \left[\frac{x_k}{2^{n-m}}\right] \tag{2}$$

which, using Fact 1 in the Appendix, can be written as

$$y_k = \left[\frac{kw}{2^{n-m}}\right] \bmod 2^m. \tag{3}$$

Using Fact 2 in the Appendix, the difference $d_k = (y_k - y_{k-1}) \bmod 2^m$ can be expressed as

$$d_k = \left(\left[\frac{kw}{2^{n-m}}\right] - \left[\frac{(k-1)w}{2^{n-m}}\right]\right) \bmod 2^m \tag{4}$$

[7]See Definition 2.1.

[8]Considered as the functions $x_{k(t)}$ and $y_{k(t)}$ of continuous-time t.

[9]Considered as the functions $d_{k(t)}$ and $\delta_{k(t)}$ of continuous-time t.

and, by its definition, it is

$$\delta_k = \begin{cases} d_k, & \text{if } d_k > 0 \\ 2^m, & \text{otherwise} \end{cases}. \tag{5}$$

Remark: Although one may be tempted to remove the modulo operation from (4), this cannot be done. For example, for $n = 4$, $m = 2$, $w = 14$, and $k = 4$, it is

$$\left\lceil \frac{kw}{2^{n-m}} \right\rceil - \left\lceil \frac{(k-1)w}{2^{n-m}} \right\rceil = 4 = 2^m.$$

Remark: The reader familiar with DDS notices that (1) and (2) are identical to the ones modeling the accumulator's value and the truncated phase value inputted into the sinusoidal lookup table of DDS. There is, however, a fundamental difference between the operation of the FA and that of the DDS. In DDS, the discrete-time index k updates based on the clock, i.e., linearly to the continuous-time t. In contrast, the discrete-time index k of FA updates irregularly, in a nonlinear way w.r.t. the continuous-time t, for most values of w [see Figs. 3(f), 4(f), and 5(f)].

3) Definitions of Periodicity:

Definition 2.1: Let a_k, $k = 0, 1, 2, \ldots$, be a discrete-time sequence. We say that a positive integer M is a *period* of sequence $\{a_k\}$ if, for every $k = 0, 1, 2, \ldots$, it is $a_k = a_{k+M}$. The smallest period of $\{a_k\}$ is called the (discrete-time) *fundamental period* of it.

As shown in Lemma 7.1 in the Appendix, the fundamental period of a sequence $\{a_k\}_{k=0}^{\infty}$ divides every other period of it. This is used in a following section to find the fundamental period of the output signal v. Also, Definition 2.1 and Lemma 7.1 are extended accordingly for the case that index k starts from a nonzero integer value.

We define the *continuous-time fundamental period* of a function $f : [0, \infty) \to \mathcal{R}$ accordingly.

III. Discrete-Time Periods

The first step in investigating the behavior of the FA synthesizer is to derive the periods and periodic patterns of its signals. This is done in the discrete-time domain first using the discrete-time index k that counts the rising edges (or spikes) of signal $s(t)$. Later, the (real) time distances between the rising edges (or spikes) are derived, and the derivations in this section are converted into ones on the (real) continuous-time domain.

Every periodic signal in a synthesizer can potentially introduce spurious frequency components at the output. This can directly be done or through leakage via direct coupling or supporting circuits such as the power supply network.

The periodic behaviors of the state sequences $\{x_k\}$ and $\{y_k\}$ and those of the difference sequences $\{d_k\}$ and $\{\delta_k\}$ are studied. Throughout this section, we assume that the frequency word w is restricted within $0 < w < 2^n$.

A. Fundamental Discrete-Time Period of the State Sequence $\{x_k\}$ and the Reachable State Set

The state sequence $\{x_k\}$ of the FA generates all other signals. The period and the reachable set of it are derived in this section.

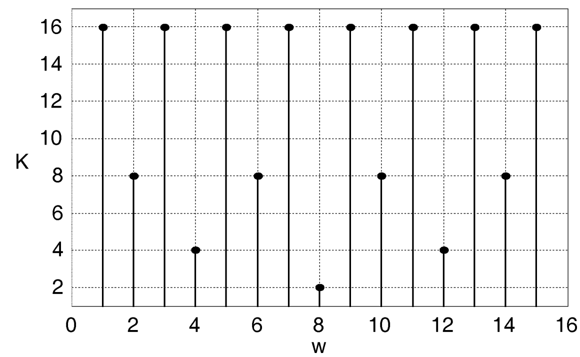

Fig. 6. Fundamental discrete-time period K of the state sequences for $n = 4$ and $w = 1, 2, \ldots, 15$.

Theorem 3.1: The fundamental discrete-time period of the state sequence $\{x_k\}$ is

$$K = \frac{2^n}{\gcd(w, 2^n)} \tag{6}$$

and K is an *even* integer. ■

Example 3.1: Fig. 6 shows the period K of the state sequence $\{x_k\}$ when $n = 4$ and the frequency word w ranges from 1 to 15. The graph is symmetric w.r.t. the center $w = 8$, and the maximum period is attained if and only if w is odd.

Proof of Theorem 3.1: Since $x_{k+a} = x_k$ if and only if $(aw) \bmod 2^n = 0$ from Fact 2 in the Appendix, the fundamental period K of the state sequence $x_k = (kw) \bmod 2^n$, $k = 0, 1, 2, \ldots$, is the smallest positive integer solution of the Diophantine equation [46]

$$(Kw) \bmod 2^n = 0.$$

From the definition of the modulo function, we have that $(Kw) \bmod 2^n = 0$ if and only if there exists an integer μ such that $Kw = \mu 2^n$, or equivalently, after dividing by $\gcd(w, 2^n)$

$$K \frac{w}{\gcd(w, 2^n)} = \mu \frac{2^n}{\gcd(w, 2^n)}. \tag{7}$$

Note that both $w/\gcd(w, 2^n)$ and $2^n/\gcd(w, 2^n)$ are integers, and since they share no common divisor other than ± 1, from (7), we get that $2^n/\gcd(w, 2^n)$ must divide K, i.e., $K = \nu 2^n/\gcd(w, 2^n)$, for some integer ν. The latter is a solution of $(Kw) \bmod 2^n = 0$ for every integer ν, and $\nu = 1$ gives the smallest positive one.

Finally, since $0 < w < 2^n$, it must be $\gcd(w, 2^n) < 2^n$, and since 2^n has no other divisors than 2, it must be $\gcd(w, 2^n) = 2^r$ for some integer r, $0 \le r \le n - 1$. Therefore, $K = 2^n/\gcd(w, 2^n)$ is even. ■

Theorem 3.1 has extensions with regard to the initial condition x_0 and the set of values of the state sequence $\{x_k\}$, i.e., the state reachable set.

Theorem 3.2: The state sequence with nonzero initial condition $x_0 = q$, i.e., $x_k = (kw + q) \bmod 2^n$, $k = 0, 1, 2, \ldots$, has fundamental discrete-time period K given by (6) as well. ■

The proof is almost identical to that of Theorem 3.1.

Corollary 3.1: Within every fundamental discrete-time period K, the state sequence takes all values in the set

$$\mathcal{X} = \{k2^r \,|\, k = 0, 1, \ldots, 2^{n-r} - 1\} \tag{8}$$

exactly once, and only them, where integer r is defined by $2^r = \gcd(w, 2^n)$. Moreover, the truncated state sequence $\{y_k\}$ takes all values in the set

$$\mathcal{Y} = \left\{(k2^r) \operatorname{div} 2^{n-m} \,|\, k = 0, 1, \ldots, 2^{n-r} - 1\right\} \tag{9}$$

at least once. ■

Proof: Let integer $r \geq 0$ be defined by $2^r = \gcd(w, 2^n)$. Then, $w = a2^r$ and $\gcd(a, 2^{n-r}) = 1$ for some positive odd integer a. Therefore, from Theorem 3.1, the sequence $\tilde{x}_k = (ka) \bmod 2^{n-r}$, $k = 0, 1, 2, \ldots$, has fundamental period $\tilde{K} = 2^{n-r}$. Moreover, $0 \leq \tilde{x}_k \leq 2^{n-r} - 1$, and thus, the sequence $\{\tilde{x}_k\}$ takes all values in $\{0, 1, \ldots, 2^{n-r} - 1\}$ exactly once, within every fundamental period of it. Equality

$$(kw) \bmod 2^n = 2^r \left((ka) \bmod 2^{n-r}\right)$$

gives $x_k = 2^r \tilde{x}_k$, which, along with the fact that $K = \tilde{K}$, conclude the proof of the first part. Equation (9) is an immediate consequence of (8). ■

The extension of Corollary 3.1 in the case of nonzero initial condition is straightforward. Similar classification of the reachable set of states of $\{x_k\}$ has been used to derive the spectrum of DDS [41], [42], [44].

Example 3.2: Consider the case in Fig. 5 with $n = 4, m = 2$, and $w = 15$. As shown in Fig. 5(b), the state sequence takes all values from 0 to 15, and the truncated sequence in Fig. 5(c) takes all values from 0 to 3, confirming Corollary 3.1.

In contrast, if $n = 4$, $m = 2$, and $w = 12$, then $r = 2$, and the state reachable set is $\mathcal{X} = \{0, 4, 8, 12\}$. The truncated sequence takes all values in $\mathcal{Y} = \{0, 1, 2, 3\}$.

B. Fundamental Discrete-Time Period of the Truncated State Sequence $\{y_k\}$

The state sequence $\{x_k\}$ having fundamental discrete-time period K directly implies that the truncated sequence $\{y_k\}$ is also K-periodic. However, this does not mean that K is also the fundamental period of $\{y_k\}$, i.e., it may have a shorter one. The following theorem resolves this question.

Note that, if $m = n$, then $x_k = y_k$ for all $k = 0, 1, \ldots$, and thus, the above question has a trivial answer.

Thus, in this section, we assume that $1 \leq m < n$.

Theorem 3.3: The fundamental period of the truncated state sequence $\{y_k\}$ is equal to that of the state sequence $\{x_k\}$ given by (6). ■

We use the following lemma to prove the theorem.

Lemma 3.1: If u is a positive integer such that $u \bmod 2^n > 0$, then there exists a positive integer r for which $((ru) \bmod 2^n) \operatorname{div} 2^{n-m} > 0$. ■

Proof: Without loss of generality, we assume that $u < 2^n$. If $u \geq 2^{n-m}$, then we can choose $r = 1$. Now, we suppose that $u < 2^{n-m}$, and we show that the positive integer[10] $r = \lceil 2^{n-m}/u \rceil$ has the desirable property. Note that, by the definition of the ceiling function, there exists ε such that $2^{n-m}/u + \varepsilon = r$ and $0 \leq \varepsilon < 1$. Multiplying the equality by u gives $2^{n-m} + \varepsilon \cdot u = r \cdot u$, and thus, $2^{n-m} \leq r \cdot u = 2^{n-m} + \varepsilon \cdot u < 2^{n-m+1}$ because of $0 \leq \varepsilon < 1$ and our assumption that $u < 2^{n-m}$. Moreover, since we have assumed that $m \geq 1$, we conclude that $2^{n-m} \leq r \cdot u < 2^n$, which implies that $((ru) \bmod 2^n) \operatorname{div} 2^{n-m} \geq 1$. ■

Proof of Theorem 3.3: Suppose that the truncated state sequence $\{y_k\}$ has fundamental period $P < K$, then since $x_0 = 0$, $y_0 = 0$, and therefore, $y_{\nu P} = 0$ for all positive integers ν. Suppose now that $x_P = (Pw) \bmod 2^n > 0$, then by Lemma 3.1, there exists a positive integer r for which $((rPw) \bmod 2^n) \operatorname{div} 2^{n-m} > 0$, and thus, it is $y_{rP} > 0$, which is a contradiction. Therefore, it must be $(Pw) \bmod 2^n = 0$, and thus, P is also a period[11] of $\{x_k\}$. Since K is the fundamental period of $\{x_k\}$, Lemma 7.1 in the Appendix implies that $P = \rho K$ for some positive integer ρ, which leads to a contradiction because of our assumption that $P < K$. ■

The statement of Theorem 3.3 is valid for nonzero initial condition $x_0 > 0$ as well. The proof is similar to that of Theorem 3.3.

Example 3.3: The result of Theorem 3.3 is confirmed by the plots of $\{x_k\}$ and $\{y_k\}$ in Figs. 3–5. In all of them, it is $K = 16$ (attention should be paid in extracting the periods w.r.t. the values of discrete-time k in parts (f) and not the continuous-time values of x-axis).

Remark: One can observe the similarity between the expressions given in Theorems 3.1–3.3 and those describing the operation of the DDS [38]. Note, however, that there is a fundamental qualitative difference between the FA and the DDS. Here, the discrete-time periods do *not* correspond to continuous-time ones in an obvious way as they do in the case of DDS. In the FA, the continuous-time it takes for index k to increase by 1, i.e., the time between any two consecutive rising edges of $s(t)$, depends on sequence y_k, which is in contrast to DDS, where the register is updated at the fixed clock rate.

C. Fundamental Discrete-Time Periods of the Difference Sequences $\{d_k\}$ and $\{\delta_k\}$

In this section, the period of the difference and modified difference sequences, i.e., $\{d_k\}$ and $\{\delta_k\}$, respectively, is derived, and important periodicity properties are extracted.

Since the output signal of the FA, i.e., $v(t)$, results from counting modulo 2 the rising edges of $s(t)$ in the D-FF, the behavior of the difference sequences, particularly that of $\{\delta_k\}$, is critical in understanding the properties of $v(t)$.

Theorem 3.4 shows that the fundamental period of the difference sequences is usually much *smaller* than that of the state sequences. Note that a smaller fundamental period of the output signal $v(t)$ is in most cases desirable since it results in less dense spurious frequency components and spurs adjacent to the carrier that are located further away from it.

Notation: The terms of all the FA's sequences depend on the frequency word w. The same is true for the fundamental periods of the sequences. Depending on the context, we may use or drop this dependence for notational convenience.

[10] By definition, the *ceiling* of a real number x, i.e., $\lceil x \rceil$, is the smallest integer h such that $x \leq h$.

[11] Note that, because of (1), $x_P = x_0$ implies that $x_{P+a} = x_a$ for all nonnegative integers a, and thus, P is a period of $\{x_k\}$ by Definition 2.1.

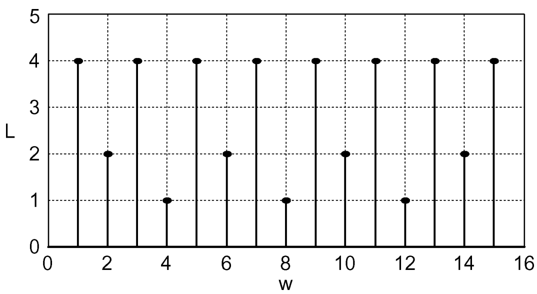

Fig. 7. Fundamental discrete-time period L of the difference sequences $\{d_k\}$ and $\{\delta_k\}$ for $n = 4$, $m = 2$, and $w = 1, 2, \ldots, 15$.

Theorem 3.4: The fundamental discrete-time period of the difference sequence $\{d_k\}$ and the modified difference sequence $\{\delta_k\}$ is

$$L = \frac{2^{n-m}}{\gcd(w, 2^{n-m})}. \tag{10}$$

■

Remark: Note that $L = 1$ means that the sequences $\{d_k\}$ and $\{\delta_k\}$ are *constant* (for all $k = 1, 2, 3, \ldots$). Note also that, if $n = m$, then from Fact 2 in the Appendix $d_k = w \bmod 2^m$, for all $k = 1, 2, 3, \ldots$, i.e., constant, which is in agreement with $L = 1$ derived from (10). Therefore, in the proof of Theorem 3.4, we consider only the case of $n \geq m + 1$.

Example 3.4: Fig. 7 shows the period L of the difference sequences $\{d_k\}$ and $\{\delta_k\}$ when $n = 4$, $m = 2$, and the frequency word w ranges from 1 to 15. The graph is symmetric w.r.t. the center $w = 8$, and the maximum period $L = 4$ is attained if and only if w is odd.

To prove Theorem 3.4, we introduce the following technical lemma.

Lemma 3.2: Let $g = \gcd(w, 2^{n-m})$, then we have the following:

A) If $0 < w < 2^{n-m-1}$, then there exist a positive integer k and an integer p such that

$$kw = 2^{n-m-1} + w - g + p2^{n-m}. \tag{11}$$

B) For all values of w, there exist a positive integer k and an integer p such that

$$kw = w + p2^{n-m}. \tag{12}$$

Proof: A) Note first that this case is feasible only when $n \geq m + 2$. Then, $0 < w < 2^{n-m-1}$ implies that integer $g = \gcd(w, 2^{n-m})$ divides both 2^{n-m-1} and w, and thus, it divides the sum $2^{n-m-1} + w - g$. Therefore, there exist integers x and y satisfying the Diophantine equation [46]

$$xw + y2^{n-m} = 2^{n-m-1} + w - g. \tag{13}$$

Moreover, (13) is also satisfied if we replace x by $x + h2^{n-m}$ and y by $y - hw$, for every integer h. For sufficiently large h, it is $x + h2^{n-m} > 0$, and thus, $k = x + h2^{n-m}$ and $p = -y + hw$ satisfy (11), as well as $k > 0$.

B) Integer $k = 2^{n-m} + 1$ is positive and satisfies (12) for $p = w$. ■

Proof of Theorem 3.4: First note that, by its definition (5), the modified difference sequence $\{\delta_k\}$ has the same fundamental discrete-time period with the difference sequence $\{d_k\}$. Recall now from (4) that (*notation*: here we consider $d_k(w)$ as a *function* of w, as well as k)

$$d_k(w) = \left(\left\lfloor \frac{kw}{2^{n-m}} \right\rfloor - \left\lfloor \frac{(k-1)w}{2^{n-m}} \right\rfloor\right) \bmod 2^m.$$

Using (10) as the definition of L, we have that

$$\frac{Lw}{2^{n-m}} = \frac{w}{\gcd(w, 2^{n-m})} \tag{14}$$

which is an integer number, since $g = \gcd(w, 2^{n-m})$ divides w, and thus

$$\left\lfloor \frac{(k+L)w}{2^{n-m}} \right\rfloor = \left\lfloor \frac{kw}{2^{n-m}} \right\rfloor + \frac{w}{\gcd(w, 2^{n-m})}$$

as well as

$$\left\lfloor \frac{(k+L-1)w}{2^{n-m}} \right\rfloor = \left\lfloor \frac{(k-1)w}{2^{n-m}} \right\rfloor + \frac{w}{\gcd(w, 2^{n-m})}.$$

Subtracting the last equation from the preceding equation and taking their modulo 2^m gives

$$d_{k+L} = d_k \tag{15}$$

which is valid for all $k = 1, 2, 3, \ldots$. Therefore, sequence d_k is periodic with period L. Now, we prove that L is the fundamental period of $\{d_k\}$.

Note first that, for every integer r, it is (*notation*: here we consider $d_k(w)$ as a *function* of w, as well as k)

$$\begin{aligned} &d_k(w + r2^{n-m}) \\ &= \left(\left\lfloor \frac{k(w + r2^{n-m})}{2^{n-m}} \right\rfloor - \left\lfloor \frac{(k-1)(w + r2^{n-m})}{2^{n-m}} \right\rfloor\right) \bmod 2^m \\ &= \left(\left\lfloor \frac{kw}{2^{n-m}} \right\rfloor - \left\lfloor \frac{(k-1)w}{2^{n-m}} \right\rfloor + r\right) \bmod 2^m \\ &= (d_k(w) + r) \bmod 2^m \end{aligned}$$

and thus

$$d_k(w + r2^{n-m}) = (d_k(w) + r) \bmod 2^m. \tag{16}$$

Equation (16) is valid for all $k = 1, 2, \ldots$ and all integers r. Since $d_k(w) \in \{0, 1, \ldots, 2^m - 1\}$, the right-hand side of (16) is the bijection $d_k(w) \mapsto (d_k(w) + r) \bmod 2^m$, and thus, sequences $\{d_k(w)\}_k$ and $\{d_k(w \bmod 2^{n-m})\}_k$ have the same fundamental period for all values of w. Therefore, we only need to examine the case

$$0 < w < 2^{n-m} \tag{17}$$

and remember to replace w by $w \bmod 2^{n-m}$ in the final expression of the fundamental period.[12] For the rest of the proof, we assume that (17) is true.

From (17), we have that $\gcd(w, 2^{n-m}) = 2^r$ and $0 \leq r \leq n - m - 1$; thus, $L/2$ is an integer number.

Suppose that sequence $\{d_k\}$ has a fundamental period P and $P < L$. Then, from Lemma 7.1 in the Appendix, we know that P must divide L, and since L is a power of 2, it is $P = 2^q$ for some nonnegative integer q. Therefore, $L/2$ is an integer

[12]Note, however, that $\gcd(w + r2^{n-m}, 2^{n-m}) = \gcd(w, 2^{n-m})$, and thus, both w and $w \bmod 2^{n-m}$ result in the same value of L in (10).

multiple of P, and thus, $L/2$ is also a period of $\{d_k\}$, i.e., $d_{k+L/2} = d_k$, for all $k = 1, 2, 3, \ldots$.

Now, note that because $w < 2^{n-m}$, the fraction $w/\gcd(w, 2^{n-m})$ is an odd integer, and thus, using (14), there is a nonnegative integer μ such that

$$\frac{Lw}{2^{n-m+1}} = \frac{w}{2\gcd(w, 2^{n-m})} = \mu + \frac{1}{2}. \quad (18)$$

Note that

$$d_{k+\frac{L}{2}} = \left(\left[\frac{kw}{2^{n-m}} + \frac{Lw}{2^{n-m+1}}\right] - \left[\frac{(k-1)w}{2^{n-m}} + \frac{Lw}{2^{n-m+1}}\right]\right) \bmod 2^m$$

which using (18) implies that

$$d_{k+\frac{L}{2}} = \left(\left[\frac{kw}{2^{n-m}} + \frac{1}{2}\right] - \left[\frac{kw-w}{2^{n-m}} + \frac{1}{2}\right]\right) \bmod 2^m. \quad (19)$$

Now, based on (17), we consider the two possible cases.

1) $0 < w < 2^{n-m-1}$. Then, from Lemma 3.2(A), there exist a positive integer k and an integer p for which (11) is true. Using these values of k and p and substituting (11) into (19), we get

$$d_{k+\frac{L}{2}} = \left(\left[1 + \frac{w-g}{2^{n-m}}\right] - \left[1 - \frac{g}{2^{n-m}}\right]\right) \bmod 2^m. \quad (20)$$

Since $1 \le g \le w < 2^{n-m-1}$, it is $[1+(w-g)/2^{n-m}] = 1$ and $[1 - g/2^{n-m}] = 0$, and thus, $d_{k+L/2} = 1$. Similarly, for the same values of k and p, and by substituting (11) into

$$d_k = \left(\left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw-w}{2^{n-m}}\right]\right) \bmod 2^m \quad (21)$$

we get

$$d_k = \left(\left[\frac{1}{2} + \frac{w-g}{2^{n-m}}\right] - \left[\frac{1}{2} - \frac{g}{2^{n-m}}\right]\right) \bmod 2^m.$$

Again, $1 \le g \le w < 2^{n-m-1}$ implies that $0 \le (w-g)/2^{n-m} < 1/2$ and $0 < g/2^{n-m} < 1/2$, and thus, $[1/2 + (w-g)/2^{n-m}] = 0$ and $[1/2 - g/2^{n-m}] = 0$, leading to $d_k = 0$. Therefore

$$1 = d_{k+\frac{L}{2}} \neq d_k = 0.$$

This contradicts the conclusion that $L/2$ is a period of sequence $\{d_k\}$, resulting from the assumption that the fundamental period P of $\{d_k\}$ is $P < L$.

2) $2^{n-m-1} \le w < 2^{n-m}$. Then, from Lemma 3.2(B), there exist a positive integer k and an integer p for which (12) is true. Using these values of k and p, and substituting (12) into (19), we get

$$d_{k+\frac{L}{2}} = \left(\left[\frac{w}{2^{n-m}} + \frac{1}{2}\right] - \left[\frac{1}{2}\right]\right) \bmod 2^m$$

which, along with the assumption on w, gives $d_{k+L/2} = 1$. In addition, substituting (12) into (21) gives

$$d_k = \left(\left[\frac{w}{2^{n-m}}\right] - [0]\right) \bmod 2^m$$

and since $w < 2^{n-m}$, we get $d_k = 0$. Therefore, we have that $1 = d_{k+L/2} \neq d_k = 0$. Again, this contradicts the conclusion that $L/2$ is a period of sequence $\{d_k\}$, resulting from the assumption that the fundamental period P of $\{d_k\}$ is $P < L$. ■

Theorem 3.4 has the following simple but important corollary, whose proof is omitted.

Corollary 3.2: If the fundamental discrete-time period is $L > 1$, then L is even.

IV. Continuous-Time Periods

Here, we use the derivations in Section III to calculate the fundamental continuous-time periods of FA's signals. We start with the difference sequence $\{d_k\}$ since it is the critical link between the discrete-time and the continuous-time behavior of the FA. Then, we calculate the periods of the state sequences. As before, we assume that $0 < w < 2^n$.

A. Fundamental Continuous-Time Periods of the Output Signal v, the Intermediate Signal s, and the Difference Sequences $\{d_k\}$ and $\{\delta_k\}$

Important properties of the difference sequence $\{d_k\}$ and the modified difference sequence $\{\delta_k\}$ have been captured in the remarks of the three Examples 2.1–2.3 in Section II. We summarize them in the following lemma without proof.

Lemma 4.1:

A) It is $d_k > 0$ if and only if the kth rising edge of signal s results in a change of value from y_{k-1} to y_k. Such a change of value corresponds to a change of the selected phase and results in a $k+1$ rising edge $d_k\Delta$ seconds after the kth one.

B) For every k such that $d_k = 0$, the time interval between the kth and $k+1$ rising edges of signal s is $T = 2^m\Delta$ seconds long. Such full-cycle (T) time intervals may be repeated, for consecutive values of k, for certain values of the frequency word w.

C) From parts (A) and (B) and the definition of δ_k, we have that the length of the continuous-time interval between the kth and the $k+1$ rising edges (we also refer to it as the kth discrete-time interval) of signal s is $\delta_k\Delta$ seconds long, for every $k = 1, 2, \ldots$. ■

The above lemma leads to the following central result for the operation of FA.

Theorem 4.1: If $L = 1$, then the difference sequences $\{d_k\}$ and $\{\delta_k\}$ and their corresponding continuous-time waveforms are *constant*. If $L > 1$, then their fundamental continuous-time period is

$$T_p = \Delta \sum_{k=1}^{L} \delta_k. \quad (22)$$

Also, the fundamental continuous-time period of signal s is given by (22), for all values of L. ■

Proof: The proof results from combining Theorem 3.4 with Lemma 4.1 and summing the continuous-time lengths of the discrete-time intervals within a fundamental discrete-time period.■

Finally, we have the following corollary.

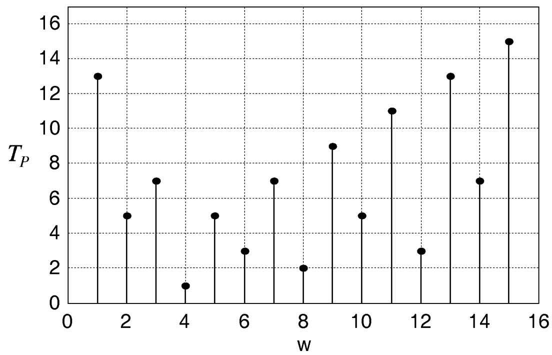

Fig. 8. Fundamental continuous-time period T_p in multiples of Δ for $n = 4$, $m = 2$, and $w = 1, 2, \ldots, 15$.

Corollary 4.1: The fundamental continuous-time period T_v of the output signal v is

$$T_v = \begin{cases} 2T_p, & \text{if } L = 1 \\ T_p, & \text{if } L > 1 \end{cases}. \tag{23}$$

Proof: The output signal v can be expressed as modulo-2 counting of the rising edges in s. If $L > 1$, then Corollary 3.2 guarantees that L is an even number. Therefore, if $v(t)$ has value $\alpha \in \{0, 1\}$ immediately after the kth rising edge of s, it will also have the same value immediately after the $k + L$ rising edge of s, and since T_p is the fundamental period of $\{\delta_k\}$ and s, it will also be that of v. If $L = 1$, the rising edges of s form a periodic sequence of fundamental continuous-time period T_p. Therefore, the fundamental period of $v(t)$ must be twice that because of the divide-by-2 action of the D-FF. ■

Our goal now is to calculate the value of T_p in closed form. The trick here is to count the number of times (values of k) within a fundamental discrete-time period L for which $d_k = 0$.

Theorem 4.2: The fundamental continuous-time period T_p in Theorem 4.1 can be expressed as

$$T_p = \Delta \cdot \begin{cases} \dfrac{2^n - (2^m - 1)w}{\gcd(w, 2^{n-m})}, & \text{if } \quad 0 < w < 2^{n-m} \\ \dfrac{w}{\gcd(w, 2^{n-m})}, & \text{if } \quad 2^{n-m} \le w < 2^n \end{cases}. \tag{24}$$

Remark: The special case $T_p = \Delta$ means that difference sequences $\{d_k\}$ and $\{\delta_k\}$ are constant.

Example 4.1: The fundamental continuous-time period T_p is shown in Fig. 8 as a function of w when $n = 4$ and $m = 2$.

Proof of Theorem 4.2: A) Consider the case $0 < w < 2^{n-m}$ first. Then, $w/2^{n-m} \le 1$, and thus, from Fact 3 in the Appendix

$$0 \le \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw}{2^{n-m}} - \frac{w}{2^{n-m}}\right] \le 1$$

which from (4) implies that

$$d_k = \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw}{2^{n-m}} - \frac{w}{2^{n-m}}\right]. \tag{25}$$

Moreover

$$d_k \in \{0, 1\}. \tag{26}$$

Consider now the discrete-time interval $k = 1, 2, \ldots, L$ of length L given by (10), corresponding to a fundamental discrete-time period of d_k. Because of (26), the cardinality of the set

$$\mathcal{S}_1 = \left\{ k \in \{1, 2, \ldots, L\} | d_k = 1 \right\} \tag{27}$$

is equal to $d_1 + d_2 + \ldots + d_L$. Using (10) and (25), and noting that $w/\gcd(w, 2^{n-m})$ is an integer

$$\sum_{k=1}^{L} d_k = \left[\frac{Lw}{2^{n-m}}\right] = \frac{w}{\gcd(w, 2^{n-m})}. \tag{28}$$

Therefore, the cardinality of the set

$$\mathcal{S}_0 = \left\{ k \in \{1, 2, \ldots, L\} | d_k = 0 \right\} \tag{29}$$

is equal to

$$L - \sum_{k=1}^{L} d_k = \frac{2^{n-m} - w}{\gcd(w, 2^{n-m})}. \tag{30}$$

Using (5), we get $\delta_k = d_k = 1$ for every $k \in \mathcal{S}_1$ and $\delta_k = 2^m$ for every $k \in \mathcal{S}_0$. Therefore

$$\sum_{k=1}^{L} \delta_k = |\mathcal{S}_1| + 2^m |\mathcal{S}_0| = \frac{2^n - (2^m - 1)w}{\gcd(w, 2^{n-m})}.$$

Using Theorem 4.1, we conclude the first part of the proof.

B1) Now, consider the case $2^{n-m} \le w \le 2^n - 2^{n-m}$. Then, $1 \le w/2^{n-m} \le 2^m - 1$ and Fact 4 in the Appendix imply that

$$1 \le \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw}{2^{n-m}} - \frac{w}{2^{n-m}}\right] \le 2^m - 1$$

and thus, from (4)

$$d_k = \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw}{2^{n-m}} - \frac{w}{2^{n-m}}\right]. \tag{31}$$

Moreover

$$1 \le d_k \le 2^m - 1. \tag{32}$$

From (32), we get $\delta_k = d_k$ for all $k = 1, 2, \ldots$, which, along with (10) and (31), imply that

$$\sum_{k=1}^{L} \delta_k = \left[\frac{Lw}{2^{n-m}}\right] = \frac{w}{\gcd(w, 2^{n-m})}.$$

Using Theorem 4.1, we conclude this part of the proof.

B2) The last case is $2^n - 2^{n-m} < w < 2^n$, which implies that $2^m - 1 < w/2^{n-m} < 2^m$, which, along with Fact 4 in the Appendix, gives

$$2^m - 1 \le \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{kw}{2^{n-m}} - \frac{w}{2^{n-m}}\right] \le 2^m.$$

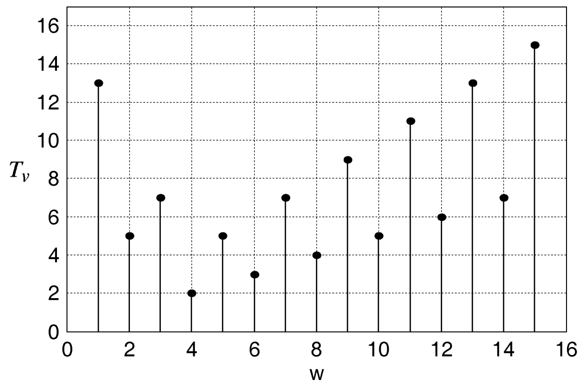

Fig. 9. Fundamental continuous-time period T_v in multiples of Δ for $n = 4$, $m = 2$, and $w = 1, 2, \ldots, 15$.

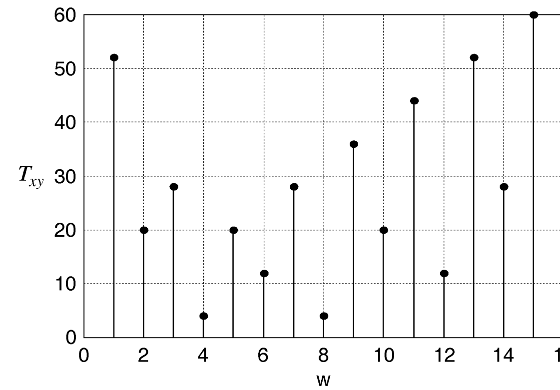

Fig. 10. Fundamental continuous-time period T_{xy} in multiples of Δ for $n = 4$, $m = 2$, and $w = 1, 2, \ldots, 15$.

Again, from the definition (4) of d_k, it is

$$d_k = \begin{cases} 2^m - 1, & \text{if } \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{(k-1)w}{2^{n-m}}\right] = 2^m - 1 \\ 0, & \text{if } \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{(k-1)w}{2^{n-m}}\right] = 2^m. \end{cases}$$

Therefore, from the definition of δ_k, we get

$$\delta_k = \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{(k-1)w}{2^{n-m}}\right]$$

for all $k = 1, 2, 3, \ldots$, which, as in case (B1), implies that

$$\sum_{k=1}^{L} \delta_k = \frac{w}{\gcd(w, 2^{n-m})}$$

and, with the use of Theorem 4.1, concludes the proof. ■

An immediate consequence of the proof of Theorem 4.2 is shown in the following corollary.

Corollary 4.2: If $2^{n-m} \leq w < 2^n$, then for all $k = 1, 2, \ldots$

$$\delta_k = \left[\frac{kw}{2^{n-m}}\right] - \left[\frac{(k-1)w}{2^{n-m}}\right]. \tag{33}$$

■

The combination of Corollary 4.1 and Theorem 4.2 results in the following explicit expressions for the fundamental continuous-time period of the output signal v.

Corollary 4.3: The fundamental continuous-time period of the output signal v is

$$T_v = \Delta \cdot \begin{cases} \frac{2^n - (2^m - 1)w}{\gcd(w, 2^{n-m})}, & \text{if } 0 < w < 2^{n-m} \\ \frac{w}{2^{n-m-1}}, & \text{if } 2^{n-m} \leq w < 2^n \text{ and } L = 1 \\ \frac{w}{\gcd(w, 2^{n-m})}, & \text{if } 2^{n-m} \leq w < 2^n \text{ and } L > 1. \end{cases} \tag{34}$$

Proof: Note that $0 < w < 2^{n-m}$ implies that $L > 1$ and that $L = 1$ if and only if $\gcd(w, 2^{n-m}) = 2^{n-m}$. Then, use Corollary 4.1 and Theorem 4.2. ■

Example 4.2: The fundamental continuous-time period T_v is shown in Fig. 9 as a function of w when $n = 4$ and $m = 2$. It should be compared with Fig. 8 with reference to the values of L in Fig. 7.

B. Fundamental Continuous-Time Period of the State Sequences $\{x_k\}$ and $\{y_k\}$

The periods of $\{x_k\}$ and $\{y_k\}$ are important to know because the two signals can introduce spurs at the output of the FA through parasitic coupling. As shown below, their period may be significantly larger than that of the output v, and thus, any leakage of them to the output may result in much denser spurious spectral components than ideally expected. From Theorem 3.3, we know that the two sequences have the same fundamental continuous-time period.

Lemma 4.2: The fundamental continuous-time period of the state and truncated state sequences, i.e., $\{x_k\}$ and $\{y_k\}$, respectively, is

$$T_{xy} = 2^m \Delta \cdot \begin{cases} \frac{2^n - (2^m - 1)w}{\gcd(w, 2^n)}, & \text{if } 0 < w < 2^{n-m} \\ \frac{w}{\gcd(w, 2^n)}, & \text{if } 2^{n-m} \leq w < 2^n \end{cases}. \tag{35}$$

Proof: The fundamental discrete-time period K of $\{x_k\}$ and $\{y_k\}$ is necessarily a period of $\{d_k\}$ and $\{\delta_k\}$ as well. Since L is the fundamental discrete-time period of $\{d_k\}$ and $\{\delta_k\}$, Lemma 7.1 in the Appendix implies that K/L is an integer. Therefore, the length of a fundamental continuous-time period of $\{x_k\}$ and $\{y_k\}$ is equal to the total length of K/L fundamental continuous-time periods of $\{\delta_k\}$, i.e.,

$$T_{xy} = \frac{K}{L} T_p.$$

Combining (6) and (10) gives

$$\frac{K}{L} = \frac{2^m \gcd(w, 2^{n-m})}{\gcd(w, 2^n)}$$

which leads to (35) using (24). ■

Example 4.3: The fundamental continuous-time period T_{xy} is shown in Fig. 10 as a function of w when $n = 4$ and $m = 2$. It should be compared with Fig. 9.

V. Average Frequency of the Output Signal v

The *average frequency* f_{av} of the output signal v is defined as the number of cycles of v within a fundamental continuous-time period T_v of it divided by T_v [10]. As it is shown in [45], under certain conditions on the parameters n, m, and w, the output

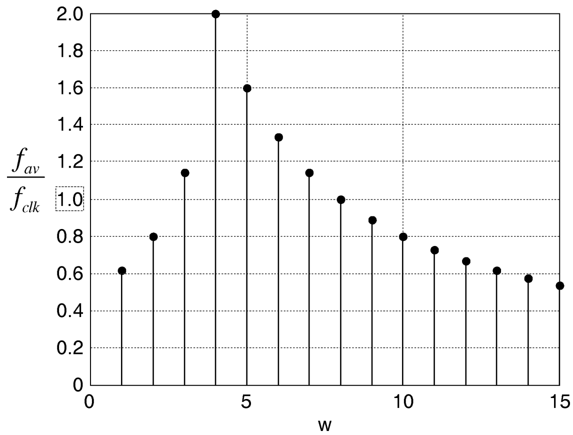

Fig. 11. Average frequency (normalized w.r.t. f_{clk}) when $n = 4$, $m = 2$, and $w = 1, 2, \ldots, 15$.

signal is regular enough to resemble a (ideal) periodic square wave of period f_{av} in both the time and frequency domains.

Corollary 5.1: The average frequency of the output signal v is[13]

$$f_{\mathrm{av}} = f_{\mathrm{clk}} \cdot \begin{cases} \frac{1}{2}, & \text{if} \quad w = 0 \\ \dfrac{2^{n-1}}{2^n - (2^m - 1)w}, & \text{if } \ 0 < w < 2^{n-m} \\ \dfrac{2^{n-1}}{w}, & \text{if } \ 2^{n-m} \le w < 2^n. \end{cases} \tag{36}$$

Proof: If $w = 0$, then $s(t) = \phi_{y_0}(t)$ all the time (see Fig. 1), and since the D-FF divides the frequency by 2, the frequency of v is $f_{\mathrm{clk}}/2$. Now, assume that $w > 0$, and note first that signal s has L rising edges within a period T_p, as given by (24). A cycle of v corresponds to two rising edges of s; thus, we have the following: 1) If $L = 1$, there is exactly one cycle every $2T_p$ seconds (note that, in this case, from (23), we have $T_v = 2T_p$); and 2) if $L > 1$, there are exactly $L/2$ cycles every T_p seconds (here, it is $T_p = T_v$, and L is even from Corollary 3.2). Therefore, in both cases, it is $f_{\mathrm{av}} = L/(2T_p)$. Using (10) and (24), and $2^m \Delta = T = 1/f_{\mathrm{clk}}$, we derive (36). ■

Note also that, since the rising-edge-triggered D-FF is a frequency divider-by-2, the average frequency of signal $s(t)$ is exactly $2f_{\mathrm{av}}$ (counting spikes as pulses).

Example 5.1: The average frequency is shown in Fig. 11 as a function of w when $n = 4$ and $m = 2$.

The following corollary is a direct consequence of Corollary 5.1 and is stated without proof.

Corollary 5.2: A) The minimum and maximum values of the average output frequency are

$$f_{\mathrm{av}}^{\min} = \frac{1}{2} f_{\mathrm{clk}} \tag{37}$$

and

$$f_{\mathrm{av}}^{\max} = 2^{m-1} f_{\mathrm{clk}} \tag{38}$$

which are achieved for $w = 0$ and $w = 2^{n-m}$, respectively. B) For $w = 0$, the output signal $v(t)$ is the 50% duty-cycle ideal square waveform. C) When $2^{n-m} \le w < 2^n$, the minimum frequency is $f_{\mathrm{clk}}/2/(1 - 2^{-n})$, the maximum is given by (38), and the frequency step is

$$\delta f_{\mathrm{av}} \approxeq \frac{f_{\mathrm{av}}^2}{2^{n-1} f_{\mathrm{clk}}}. \tag{39}$$

D) For every frequency word w_1 such that $0 < w_1 < 2^{n-m}$, the frequency word $w_2 = 2^n - (2^m - 1)w_1$ satisfies $2^{n-m} \le w_2 < 2^n$, and both frequency words result in the same value of f_{av}. ■

Note that part (D) of Corollary 5.2, along with the discussion in Section II about the irregularity of $v(t)$ when $0 < w < 2^{n-m}$, indicates that, for most applications, we should prefer using frequency words in the range $2^{n-m} \le w_2 < 2^n$, and that this choice provides all possible values of f_{av} that the FA can generate.[14]

Based on the discussion in the proof of Corollary 5.1 and using the result of Corollary 4.1, we can establish Corollary 5.3. Recall that the fundamental period of $v(t)$ is T_v, and therefore, the fundamental frequency of $v(t)$ is

$$f_{v_f} = \frac{1}{T_v} \tag{40}$$

Corollary 5.3: The average frequency f_{av} of the output signal $v(t)$ is equal to the N_{av} harmonic of $v(t)$, i.e.,

$$f_{\mathrm{av}} = N_{\mathrm{av}} f_{v_f} \tag{41}$$

where

$$N_{\mathrm{av}} = \begin{cases} 1, & \text{if } \ L = 1 \\ L/2, & \text{if } \ L > 1 \end{cases}. \tag{42}$$

Equivalently: There are N_{av} cycles of $v(t)$ within every fundamental continuous-time period T_v of it. ■

Typically, one expects to observe spectral components (spurs) at the output of the FA at the harmonics $q f_{v_f}$, $q = 0, 1, 2, \ldots$, of $v(t)$. Since f_{av} is the N_{av} harmonic, the spurs closest to f_{av} are $f_{\mathrm{av}} \pm f_{v_f}$. In particular, in RF applications, these spurs are unwanted, and one may want to remove them by filtering. The smaller the relative difference

$$\left| \frac{f_{\mathrm{av}} - \left(f_{\mathrm{av}} \pm f_{v_f}\right)}{f_{\mathrm{av}}} \right| = \frac{1}{N_{\mathrm{av}}}$$

is, the more difficult this task is.

VI. Conclusion

The FA frequency synthesizer with n-bit register, 2^m input clock phases, and frequency word w has been mathematically modeled.

The fundamental continuous-time period T_v of the output signal has been derived in closed form and shown to be a nonmonotonic function of w. The fundamental continuous-time period T_{xy} of the register's value (state) and the truncated register's value driving the MUX has also been derived in closed form and found to be significantly larger than T_v (up to 2^m times) for most values of w.

[13]We make the exception to consider the case $w = 0$ here.

[14]The FA with $0 < w < 2^{n-m}$ may be useful as a pattern generator rather than a frequency generator.

The average output frequency $f_{\rm av}$ has been derived in closed form. Its minimum value is $f_{\rm av}^{\min} = f_{\rm clk}/2$ and its maximum value is $f_{\rm av}^{\max} = 2^{m-1} f_{\rm clk}$. For frequency word $w \geq 2^{n-m}$, the average output frequency is $f_{\rm av} = (2^{n-1}/w) f_{\rm clk}$ and, therefore, monotonically decreasing with w. For most values of the frequency word $w < 2^{n-m}$, the output waveform is very irregular, and the average output frequency is an increasing function of w.

The simplicity and fully digital architecture of the FA, along with its good output-period resolution and range, makes it a good candidate for digital circuits applications that are tolerant to bounded timing irregularities.

Part II of the paper [45] characterizes the timing structure of the output waveform and presents analytical bounds of its timing irregularities and exact analytical expressions of several standard jitter metrics, as well as spectral properties of the output, including the dominance of the frequency component at the average frequency.

APPENDIX

The following five *facts* can be found in [47] and other books on discrete mathematics.

- Fact 1: Let x, a, and b be nonnegative integers and $a \geq b$, then $(x \bmod 2^a) \operatorname{div} 2^b = (x \operatorname{div} 2^b) \bmod 2^{a-b}$.
- Fact 2: Let x, y, and a be integers and $a \geq 1$, then we have that $(x \bmod a \pm y \bmod a) \bmod a = (x \pm y) \bmod a$.
- Fact 3: If $0 \leq y - x \leq 1$, then $0 \leq [y] - [x] \leq 1$.
- Fact 4: Let p and q be two nonnegative integers, then $p \leq y - x \leq q$ implies that $p \leq [y] - [x] \leq q$.
- Fact 5: If $x \leq y$, then $[y - x] \leq [y] - [x] \leq \lceil y - x \rceil$.

Lemma 7.1: The fundamental period of a sequence $\{a_k\}_{k=0}^{\infty}$ divides every other period of it.

Proof: Let the sequence $\{a_k\}_{k=0}^{\infty}$ have fundamental period M and another period N. Let $g = \gcd(M, N)$. From [46], we know that there exist integers α and β satisfying the Diophantine equation $\alpha M + \beta N = g$. Since $g > 0$, at least one of αM and βN is nonnegative. Let us assume that $\alpha M \geq 0$, then for every $k \geq 0$, we have that $a_k = a_{k+\alpha M}$ and $a_{k+\alpha M+\beta N} = a_{k+\alpha M}$, which imply that $a_{k+g} = a_k$ (note that all indexes are nonnegative). The same is true if we assume that $\beta N \geq 0$. Therefore, g is also a period of $\{a_k\}$, and by its definition, it is $g \leq M$. However, M is the smallest period of $\{a_k\}$, and thus, $g = M$. By definition of g, we also get that M divides N. ■

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