Energy Reduction in VLSI Computation Modules: An Information-Theoretic Approach

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Abstract*—We consider the problem of reduction of computation cost by introducing redundancy in the number of ports as well as in the input and output sequences of computation modules. Using our formulation, the classical *"communication scenario"* is the case when a computation module has to recompute the input sequence at a different location or time with high fidelity and low bit-error rates. We then consider communication with different computational cost objective than that given by bit-error rate. An example is communication over deep submicrometer very-large scale integration (VLSI) buses where the expected energy consumption per communicated information bit is the cost of computation. We treat this scenario using tools from information theory and establish fundamental bounds on the achievable expected energy consumption per bit in deep submicrometer VLSI buses as a function of their utilization. Some of our results also shed light on coding schemes that achieve these bounds. We then prove that the best tradeoff between the expected energy consumption per bit and bus utilization can be achieved using codes constructed from typical sequences of Markov stationary ergodic processes. We use this observation to give a closed-form expression for the best tradeoff between the expected energy consumption per bit and the utilization of the bus. This expression, in principle, can be computed using standard numerical methods. *The methodology developed here naturally extends to more general computation scenarios.

***Index Terms*—Activity, buses, coding, deep submicrometer, digital circuits, energy, entropy, Markov process, power reduction, stationary process, transition, very-large scale integration (VLSI).**

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I. Introduction

IT IS fair to say that computation is the most central problem in applied science and engineering and almost all problems in these fields are concerned with computing certain desired parameters. In this light, much effort has been dedicated to designing circuitry and algorithms for computation. The designer of such circuits or algorithms always seeks to compute the desired parameters while optimizing certain other objective functions such as the cost of circuitry, the energy consumption, and the time required to do the computation.

We model a general computation device by a "*black box*," $\mathcal{B}$. The input to the device at time t is a vector I_t of n input bits and the output is a vector O_t of m bits. Assuming a causal model, depending on the machine $\mathcal{B}$, a sequence of input vectors I_t, $t = 1, 2, \ldots, T$ and a sequence of output vectors O_t, $t = 1, 2, \ldots, T$, the device $\mathcal{B}$ at time t has some computation cost (such as time, energy, etc.). We assume that the computation cost is a known function of the input vector sequences.

In the above example, the input is "*uncoded*," meaning that all input sequences of vectors are possible. Thus, we can pose the following fundamental problem: Suppose that we can make a second device with larger input and output ports than the minimum required for the computation. Can we use the second device and exploit the redundancy using coding, so that the resulting computation cost is less than that of the original device?

Naturally, using such a coding scheme implies that the input and output sequences can be mapped into low-cost sequences of the second device. The above problem is fundamental and worth answering. However, cost models for most computation circuits are hard to find. In the absence of a general model, in this paper, we restrict our attention to circuits with known computation cost functions. *The methods that we develop though can be applied to a very general class of circuits.*

One interesting example is the case when the output vector is identical to the input vector. This is the case when the computing module reproduces the input vector with high fidelity at another location or time. This is precisely the "*communication scenario*." Communication is simply a small subset of the computational problems which can be dealt with using our method.

For some classical channels, coding for reducing the probability of error in this scenario has a well-established history. In fact, if the cost of computation is the bit-error probability, much effort has been dedicated to decrease this cost using error-correction codes in the past 50 years.

In this paper, we consider the communication scenario. However, we assume that we have an error-free communication channel. Instead of using the probability of error in this case as

the cost of communication, we consider the power consumption of the communication devices as our objective function. The motivation to consider this problem stems from a familiar scenario in digital circuit design. Digital circuits are composed of different operational units that execute specific instructions. The units need to communicate in certain combinations and interchange data as well as instruction sequences. For this reason, there is a communication network between the units consisting of individual paths. In many cases, these paths are sets of identical channels laid out in parallel forming a *bus*. The structure of a bus is, in general, simple as it is composed only of drivers (transmitters), receivers, and wires between them carrying digital signals. Every bus has a certain number of parallel channels (the wires) which we also call *lines*. Although the bus is by nature a continuous (time and voltage) electrical system, certain timing assumptions allow us to think of it as a discrete value–discrete time system. Under these assumptions, at each time moment, every line carries a bit of information. Thus, we define a *bus* as a communication device where the input vector is reconstructed at the output with very high fidelity.

When communicating over a bus, a cost for information transmission has to be paid. For every single bit transmitted through the bus, the receiver and transmitter consume a certain amount of electric energy. Energy is consumed when there is a change in the input sequence of the bus.

We also refer to the input vector at time t as the *state* of the bus at time t. We find it convenient to use both names *input* and *state* for one object, the vector. The reason is that although the bus does not have memory as a computation device (in contrast to a general *finite-state machine*), it does have memory (and so state) when it comes to energy consumption. With a minor abuse of the definition, we say that the bus has a *transition* from a state S_t at time t to another state S_{t+1} at time $t+1$, even when $S_{t+1} = S_t$.

We write S_t for the bus input vector at time t, where $S_t = (s_t^1, s_t^2, s_t^3, \ldots, s_t^n)$ and $s_t^1, \ldots, s_t^n$ are the bits transmitted through lines $1, 2, \ldots, n$, respectively, at time t. By default, the value of a bit is a binary number, 0 or 1. For convenience though, we regard 0 and 1 as *reals* when we use the operators $+$ and $-$ (in contrast to the binary addition $\oplus$). This is in particular the case of energy expression (1).

The energy cost of the transition from state S_t to state S_{t+1} can be computed explicitly (see [17]) and is equal to

$$\mathcal{E}(S_t \to S_{t+1}) = E_0(S_{t+1} - S_t)\mathcal{A}(S_{t+1} - S_t)' \qquad (1)$$

where V' is the *transpose* of vector V and matrix $\mathcal{A}$ is shown in (2) at the bottom of the page. The real parameter λ is nonnegative and depends on the physical properties of the lines such as geometry, size, and distances between them as well as the type of technology used in the manufacturing process of the bus. The constant E_0 also depends on the technology and the physical design of the bus. For the obsolete *non-submicrometer* (NSM) technologies λ is practically zero and $\mathcal{A}$ reduces to a scalar matrix. For modern *deep submicrometer technologies* (DSM) λ can be as high as 8 (for example in 0.13-μm technologies). The total energy dissipation corresponding to a sequence $S_1, S_2, S_3, \ldots, S_T$ is given by $\sum_{t=1}^{T-1} \mathcal{E}(S_t \to S_{t+1})$. We can also consider the transition to S_1 from the initial state S_0 that has cost equal to $\mathcal{E}(S_0 \to S_1)$.

In the spirit of the above, it is now natural to ask the following fundamental question.

The Coding Problem: Is it possible to reduce the expected energy per transmitted bit by adding more lines in the bus? If so, what are the achievable limits and the coding schemes to be used?

The key element here is that we add extra lines in the bus while the data stream, that has to be transmitted remains unchanged. By doing this, we pay an extra area cost on the microchip but we also get a communication channel of higher capacity. Therefore, the question stated above can be rephrased as follows: what is the relation between additional capacity of the channel and possible energy reduction?

Now note that redundancy of the capacity of the channel (bus) can be ensued not only by bus expansion but also by data rate reduction. From an application point of view, this is a completely different problem. From a theoretical formulation point of view, this is exactly the same question as that we asked before, that is: what is the "best" relation between the rate information is being transmitted and the rate energy is being consumed?

There are some places in microprocessor architecture where redundancy is already present because of highly "temporally" or/and "spatially" correlated data. A particular example is address buses where the amount of information transmitted each time is less than a bit while the size of the bus is 8 or 16. In such cases, from an information-theoretic perspective, there is a tremendous amount of intrinsic redundancy that can be exploited to reduce power consumption. Again, the question is how much energy reduction is possible?

Some aspects of this problem have been studied for NSM buses ($\lambda = 0$) before. A relation between the bit rate and energy consumption as well as some coding schemes for $\lambda = 0$

$$\mathcal{A} = \begin{pmatrix} 1+2\lambda & -\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\lambda & 1+2\lambda & -\lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\lambda & 1+2 \end{pmatrix}. \qquad (2)$$

have been presented in [12]. Related work involving both theoretical and practical aspects of the problem of estimating energy consumption and the problem of coding design for energy reduction has been presented, for example, in [2], [5], [11], [13], [22]–[24], [16], [6], [14], [15], and [18]–[20]. It is interesting to mention that almost all the proposed practical coding schemes have memory one, in the sense that if $\hat{S}_t$ is the vector transmitted at time t (through the expanded bus), then $\hat{S}_t$ depends only on $\hat{S}_{t-1}$ and the data vector S_t at time t. In general, a practical coding scheme must introduce minimum possible complexity and transmission delay to the already very complex (see [1]) modern microprocessor architectures. Nonetheless, there are certain places in the microprocessors where significant delay can be tolerated allowing for more sophisticated coding.

Note that in the case where $\lambda = 0$ the cost function of the general bus with n lines reduces into the sum $\sum_{j=1}^{n}(s_{t+1}^{j} - s_t^j)^2$. This format has motivated some researchers [23], [24] to design coding schemes which reduce the average Hamming distance between consecutive states of the bus.

In the case of DSM technologies ($\lambda > 0$), unlike the case of NSM ($\lambda = 0$), the above cost function has terms corresponding to interactions between the values transmitted on different lines. This makes the treatment of the above coding problem in the DSM case a more challenging task that is addressed in the present paper.

The outline of this paper is given next. In Section II, we consider the above coding problem. Using a differential coding scheme where the codewords are carefully chosen to have low Hamming weights with high probabilities, we compute a general upper bound on the minimum possible average energy consumption in a DSM bus. When $\lambda = 0$, our bound is given by an explicit simple formula and coincides with that of [12]. In this section, we also define the expected energy consumption per bit and the utilization of the bus by a stationary process.

In Section III, we provide a nonconstructive coding scheme based on the typical sequences of certain ergodic stationary Markov processes. These schemes lead to more powerful existence results and higher power reduction than those of Section II. In contrast, it is harder to compute these reductions numerically, except for buses with a small number of lines. These improvements motivate us to consider the category of stationary processes whose states correspond to the states of the bus.

We prove that for every stationary process there is a stationary ergodic Markov process of the same entropy rate (bus utilization) and less than or equal to energy per bit. As a consequence, it is shown that the minimum possible energy per bit at a given utilization of the bus is asymptotically achievable using a code whose codewords are (finite) typical sequences of an ergodic Markov process.

In Section IV, we study the properties of the minimum energy per bit as a function of the bus utilization and establish its continuity. We formulate the computation of the *minimum energy per bit* function as a convex optimization problem which we solve analytically. An explicit form of this function, that works for a very general class of cost functions (transition costs), is provided. Details of this derivation as well as some related issues can be found in [25], [26].

Conclusions and final remarks are given in Section V.

II. The Entropy Bound on Achievable Energy Reduction

In this section, we consider the bus model described in the previous section. Recall that the energy cost $\mathcal{E}(S_t \to S_{t+1})$ of the transition $S_t \to S_{t+1}$ is given by (1), where the $n \times n$ matrix $\mathcal{A}$ is given by (2). The constant E_0 depends on the technology and n is equal to the number of lines in the bus [17]. For convenience, we set $E_0 = 1$ throughout this paper. Finally, recall that the total energy required to transmit a sequence $\boldsymbol{c} = (S_1, S_2, S_3, \ldots, S_L)$ of L succesive input vectors is

$$\mathcal{E}(\boldsymbol{c}) = \sum_{t=1}^{L-1} \mathcal{E}(S_t \to S_{t+1}). \tag{3}$$

Let $\mathcal{Q} = \{0, 1\}^n$ be the set of all binary vectors of length n. For every t, the state (and input) S_t of the bus at time t is in $\mathcal{Q}$. For every $S \in \mathcal{Q}$, let $w(S)$ denote the Hamming weight of S. We will also use the notation of $S \oplus U$ for the binary sum of any two vectors $S, U \in \mathcal{Q}$. Recall that in the calculation of the transition energy $\mathcal{E}(S_t \to S_{t+1})$ using (1) we regard S_t and S_{t+1} as *real* and not binary vectors.

In order to establish achievable upper bounds on the minimum possible average energy consumed per transmitted bit, we first have to establish some technical results. We start with the following simple albeit fundamental definitions and lemma.

Definition 2.1: We define a code $\mathcal{C}$ of length L as a set of sequences $\boldsymbol{c}$ of L successive bus input vectors $\sigma_1, \sigma_2, \ldots, \sigma_L$ where $\sigma_i \in \mathcal{Q}$. The probability that the codeword $\boldsymbol{c} \in \mathcal{C}$ is transmitted is denoted by $\Pr(\boldsymbol{c})$.

Definition 2.2: The entropy *per use of the bus* of a code $\mathcal{C}$ is defined as

$$\mathcal{H}(\mathcal{C}) = -\frac{1}{L}\sum_{\boldsymbol{c}\in\mathcal{C}} \Pr(\boldsymbol{c}) \log \Pr(\boldsymbol{c}) \tag{4}$$

where $\log$ denotes the binary logarithm. This definition extends to the uncoded case where $L = 1$ and $\mathcal{C}$ is the set of all vectors in $\mathcal{Q}$, each transmitted with probability $1/2^n$.

Transmitting a codeword means that we transmit a certain sequence of L successive elements of $\{0, 1\}^n$ through the bus, so we use the bus L times.

Definition 2.3: The *expected energy consumption per use of the bus*, when applying the code $\mathcal{C}$, is defined as

$$\mathcal{E}_{\text{av}}(\mathcal{C}) = \frac{1}{L}\sum_{\boldsymbol{c}\in\mathcal{C}} \Pr(\boldsymbol{c})\mathcal{E}(\boldsymbol{c}) + \frac{1}{L}\sum_{\boldsymbol{c},\boldsymbol{c}'\in\mathcal{C}} \Pr(\boldsymbol{c})\Pr(\boldsymbol{c}')\mathcal{E}(\sigma_L \to \sigma_1') \tag{5}$$

where σ_L and σ_1' are the last and first entries (vectors) of the codewords $\boldsymbol{c}$ and $\boldsymbol{c}'$, respectively.

Note that the second term is due to the energy loss on the transition between the last state of $\boldsymbol{c}$ and the first state of $\boldsymbol{c}'$. We also agree that the energy of codewords with only one entry is zero, that is, $\mathcal{E}(\sigma) = 0$ for every $\sigma \in \mathcal{Q}$. So for codes of length one, $L = 1$, the expected energy per use of the bus becomes

$$\mathcal{E}_{\text{av}}(\mathcal{C}) = \sum_{\sigma,\,\omega \in \mathcal{C}} \Pr(\sigma)\Pr(\omega)\mathcal{E}(\sigma \to \omega).$$

In the analysis that follows, we are mostly interested in the case where the length of the code becomes arbitrarily large. In this case, the second term becomes zero and can be ignored. In particular, if we set $\mathcal{E}_{\max} = \max_{\sigma,\,\omega \in \mathcal{Q}} \mathcal{E}(\sigma \to \omega)$, there exists a nonnegative number $e(\mathcal{C})$ such that $e(\mathcal{C}) \le \mathcal{E}_{\max}$ and

$$\mathcal{E}_{\text{av}}(\mathcal{C}) = \frac{1}{L}\sum_{\boldsymbol{c} \in \mathcal{C}} \Pr(\boldsymbol{c})\mathcal{E}(\boldsymbol{c}) + \frac{e(\mathcal{C})}{L}. \tag{6}$$

Note also that, on average, we transmit $L \cdot \mathcal{H}(\mathcal{C})$ information bits per L bus uses. If no code is used, with L uses of the bus we can transmit nL information bits. A definition follows naturally.

Definition 2.4: We define *the utilization α of the bus by the code $\mathcal{C}$* to be the ratio of the expected number of information bits transmitted per use of the bus, over the number of actual bits transmitted per use of the bus without coding. It is, of course

$$\alpha = \frac{\mathcal{H}(\mathcal{C})}{n}. \tag{7}$$

Furthermore, we define the *expected energy consumption per information bit* transmitted through the bus, when using code $\mathcal{C}$, as

$$\mathcal{E}_b(\mathcal{C}) = \frac{\mathcal{E}_{\text{av}}(\mathcal{C})}{\mathcal{H}(\mathcal{C})}. \tag{8}$$

In general, we use the term *utilization* of the bus to denote the ratio of a given information rate (the expected number of bits transmitted per bus use) over the number of lines n of the bus.

Expression (8) is the ratio of the expected energy cost per bus use over the expected number of information bits transmitted per bus use.

In the case of the uncoded bus, the input vectors are regarded as uniformly distributed in the set $\mathcal{Q} = \{0, 1\}^n$ and so the individual bits transmitted are independent random variables with probability $1/2$. It can be verified directly using (1), with $E_0 = 1$, that the expected energy per use of the bus (per transition) is

$$\overline{\mathcal{E}(S_t \to S_{t+1})} = n\,\frac{(1+2\lambda)}{2}. \tag{9}$$

Throughout the paper, *overline* denotes expectation with respect to all random variables involved in the expression (unless it is stated otherwise).

Definition 2.5: The expected energy per information bit in the case of the uncoded bus will be denoted by $\mathcal{E}_u$, that is,

$$\mathcal{E}_u = \frac{(1+2\lambda)}{2}. \tag{10}$$

Lemma 2.1: Let X denote a random vector in $\mathcal{Q} = \{0, 1\}^n$ whose components are independently and uniformly distributed in $\{0, 1\}$. Let S be a given vector in $\mathcal{Q}$ and let $\overline{\mathcal{E}(X \to S \oplus X)}$ denote the expected value of the random variable $\mathcal{E}(X \to S \oplus X)$. Then

$$\overline{\mathcal{E}(X \to S \oplus X)} = 2w(S)\mathcal{E}_u \tag{11}$$

where $\oplus$ is the binary addition and $w(S)$ is the Hamming weight of vector S.

Proof: Let

$$S = (s_1, s_2, \ldots, s_n)$$
$$X = (x_1, x_2, \ldots, x_n)$$

and

$$Y = S \oplus X = (y_1, y_2, \ldots, y_n).$$

According to (1), we have to compute the expectation of $\mathcal{E}(X \to Y) = (Y - X)\mathcal{A}(Y - X)'$. Matrix $\mathcal{A}$ being given by (2) results in

$$\mathcal{E}(X \to Y) = (1+2\lambda)\sum_{i=1}^{n}(y_i - x_i)^2 - 2\lambda\sum_{i=1}^{n-1}(y_i - x_i)(y_{i+1} - x_{i+1}). \tag{12}$$

Since we deal with the components of $Y - X$ as real numbers, we have $(y_i - x_i) = (-1)^{x_i}s_i$, hence $(y_i - x_i)^2 = s_i$ and

$$(y_i - x_i)(y_{i+1} - x_{i+1}) = (-1)^{(x_i + x_{i+1})}s_i s_{i+1}.$$

Replacing them in (12) we have

$$\mathcal{E}(X \to S \oplus X) = 2\mathcal{E}_u\sum_{i=1}^{n} s_i - 2\lambda\sum_{i=1}^{n-1}(-1)^{(x_i + x_{i+1})}s_i s_{i+1}.$$

The result follows by taking expectations of both sides and using the identity $w(S) = \sum_{i=1}^{n} s_i$. □

Definition 2.6: We define a pair (α, β), where α denotes the utilization of the bus and β the expected energy per bit, to be *achievable* if and only if there exists an infinite sequence of codes with strictly increasing lengths that utilize the bus arbitrarily close to a number $\alpha_1 \ge \alpha$ and have expected energy consumption per bit that gets arbitrarily close to a number $\beta_1 \le \beta$.

Definition 2.7: We define the *limiting expected energy consumption per bit* $\mathcal{E}_b$ at utilization α of the bus $\alpha \in [0, 1]$ to be the function

$$\mathcal{E}_b(\alpha) = \inf\{\beta \,|\, \text{The pair } (\alpha, \beta) \text{ is achievable}\}. \tag{13}$$

The same symbol $\mathcal{E}_b$ has been used for both the limiting expected energy per bit as well as the expected energy per bit of a given code $\mathcal{E}_b(\mathcal{C})$. The argument will determine which one we refer to. Let $\mathcal{E}_{\max} = \max_{\sigma,\,\omega \in \mathcal{Q}} \mathcal{E}(\sigma \to \omega)$, then for any $\alpha \in [0, 1]$, the pair $(\alpha, \mathcal{E}_{\max})$ is always achievable, thus, $\mathcal{E}_b(\alpha)$ is well defined.

According to its definition, $\mathcal{E}_b(\alpha)$ is per *information* bit *the minimum possible energy we have to spend per bit transmitted*

through the bus, when the information rate is αn bits per bus use (transition).

Our first result establishes an upper bound on $\mathcal{E}_b(\alpha)$.

Theorem 2.1: Let $h^{-1}(\cdot)$: $[0, 1] \to [0, \frac{1}{2}]$ denote the inverse of the binary entropy function $h(\cdot)$ when restricted to the domain $[0, \frac{1}{2}]$. Then, for any utilization α and for any bus of size n

$$\mathcal{E}_b(\alpha) \leq \frac{2h^{-1}(\alpha)}{\alpha}\,\mathcal{E}_u. \tag{14}$$

Proof: Let U be an information source of independent vectors $U \in \mathcal{Q}$. Let the components of the vector $U = (u_1, u_2, \ldots, u_n)$ be independent and identically distributed (i.i.d.) with probability p of being one. Let $\nu = p/(1-p)$ denote the ratio of the expected number of ones to the expected number of zeros, and $w(U)$ denotes the Hemming weight $w(U) = \sum_{i=1}^{n} u_i$. Then the probability distribution of U is

$$p(U) = p^{w(U)}(1-p)^{n-w(U)} = \nu^{w(U)}/(1+\nu)^n$$

so its entropy is

$$\mathcal{H}(U) = -\sum_{U\in\mathcal{Q}} \frac{\nu^{w(U)}}{(1+\nu)^n} \log \frac{\nu^{w(U)}}{(1+\nu)^n} \tag{15}$$

$$= \frac{1}{(1+\nu)^n}\left[n \sum_{U\in\mathcal{Q}} \nu^{w(U)} \log(1+\nu) - \sum_{U\in\mathcal{Q}} w(U)\nu^{w(U)} \log \nu \right]. \tag{16}$$

Since $p(U)$ is a probability distribution, we have

$$\sum_{U\in\mathcal{Q}} \nu^{w(U)} = (1+\nu)^n$$

and

$$\sum_{U\in\mathcal{Q}} w(U)\nu^{w(U)} = n\nu(1+nu)^{n-1} \tag{17}$$

so

$$\mathcal{H}(U) = n\left(\log(1+\nu) - \frac{\nu\log\nu}{1+\nu}\right).$$

Letting $p = h^{-1}(\alpha)$ results in the entropy rate $\mathcal{H}(U) = \alpha n$. Our coding scheme is described next.

First, the initial state of the bus S_0 is assumed to be uniformly distributed in $\mathcal{Q}$ with i.i.d. components and independent of the outcomes of source U. Let $\epsilon > 0$ be given; then, at each time t, as many as $\alpha n - \epsilon$ bits can be mapped on average into outcomes of U. Such an encoding scheme exists by the Shannon source-coding theorem. Subsequently, if U_t is the outcome of the source at time t and S_{t-1} is the state of the bus at time $t-1$, then we set the state of the bus at time t to be $S_t = S_{t-1} \oplus U_t$.

Because the original state of the bus is uniformly distributed, it can be seen that at each time t, the state of the bus is also uniformly distributed in $\mathcal{Q}$. This means that we can apply Lemma 2.1 and observe that the expected energy at time t assuming that input U_t is chosen is given by

$$\overline{\mathcal{E}_{S_t}(S_t \to S_t \oplus U_t)} = 2w(U_t)\mathcal{E}_u \tag{18}$$

where the subscript indicates that the expectation is taken with respect to S_t. Recall that $p(U_t) = \nu^{w(Ut)}/(1+\nu)^n$ and so, taking expectation over S_t and U in (18) gives

$$\overline{\mathcal{E}(S_t \to S_t \oplus U_t)} = 2\mathcal{E}_u \sum_{u\in\mathcal{Q}} w(u)p(u) \tag{19}$$

$$= 2n\mathcal{E}_u \frac{\nu}{1+\nu} = 2n\mathcal{E}_u h^{-1}(\alpha) \tag{20}$$

where we have used (17). Thus, the expected energy per bit of the bus, at a utilization arbitrarily close to α, is arbitrarily close to $\frac{2h^{-1}(\alpha)}{\alpha}\,\mathcal{E}_u$. This means that the pair $\left(\alpha, \frac{2h^{-1}(\alpha)}{\alpha}\,\mathcal{E}_u\right)$ is achievable. We conclude that

$$\mathcal{E}_b(\alpha) \leq \frac{2h^{-1}(\alpha)}{\alpha}\,\mathcal{E}_u$$

as claimed. □

The bound of Theorem 2.1 is depicted in Fig. 1. The ratio $\mathcal{E}_b(\alpha)/\mathcal{E}_u$ is plotted on the vertical axis. As we can see, we can obtain significant reduction in energy consumption if we transmit data at a rate less than the maximum.

Next, we show that when the number of lines n is large it is in fact possible to achieve the above limit using uniform input distribution and by using low Hamming weight codewords. To this end, we recall the following lemma of [8].

Lemma 2.2: Let $0 \leq \nu \leq \frac{1}{2}$. Then for any n the following inequality holds:

$$\frac{2^{nh(\nu)}}{\sqrt{8n\nu(1-\nu)}} \leq \sum_{j=0}^{\lfloor \nu n \rfloor} \frac{n!}{j!(n-j)!} \leq 2^{nh(\nu)}$$

where $h(\cdot)$ is the binary entropy function.

Proof: We refer the reader to of [8, Corollary 9, Ch. 10, Sec. 11, p. 310]. □

Suppose that $0 \leq \alpha \leq 1$ is given. Now let $\nu = h^{-1}(\alpha)$ with $\nu \leq 1/2$ and consider the set $\mathcal{Q}_*$ of all elements in $\mathcal{Q} = \{0, 1\}^n$ having Hamming weight less than or equal to $n\nu$. By Lemma 2.2, the cardinality $|\mathcal{Q}_*|$ of $\mathcal{Q}_*$ satisfies

$$\frac{2^{nh(\nu)}}{\sqrt{8n\nu(1-\nu)}} \leq |\mathcal{Q}_*| \leq 2^{nh(\nu)}.$$

For large n, it can be observed from the above that $\log(|\mathcal{Q}_*|)/n \simeq h(\nu) = \alpha$. The approximation approaches the exact value as $n \to \infty$. We now state a second coding scheme.

Theorem 2.2: The bound of Theorem 2.1 can be approached using a uniform input distribution and differential encoding.

Proof: Let the initial state of the bus S_0 be uniformly distributed in $\mathcal{Q}$. Also, let U be an information source producing independent outcomes that are uniformly distributed in the set $\mathcal{Q}_*$. The entropy rate of the source is, of course, $\log(|\mathcal{Q}_*|)$. Now, suppose that the state of the bus at time $t-1$ is S_{t-1}. If U_t is the outcome of U at time t we set $S_t = S_{t-1} \oplus U_t$. Since S_0 is uniformly distributed in $\mathcal{Q}$ it can be seen that at each time t the state of the bus is also uniformly distributed in $\mathcal{Q}$. This means that we can apply Lemma 2.1 and so the expected energy at time t is given by

$$\overline{\mathcal{E}_{S_t}(S_t \to S_t \oplus U_t)} = 2w(U_t)\mathcal{E}_u \tag{21}$$

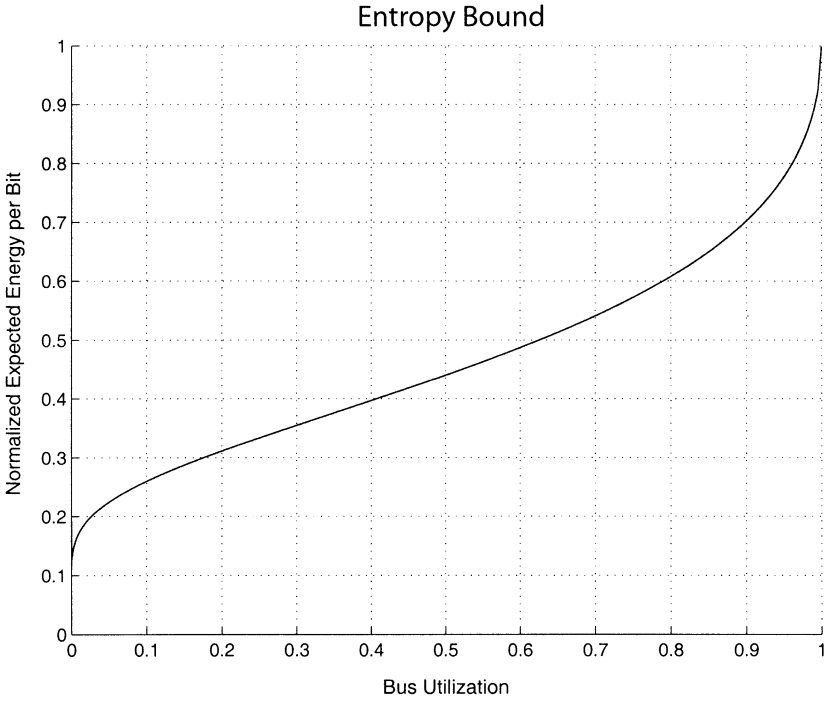

Fig. 1. The entropy bound.

where the subscript indicates that the expectation is taken over with respect to S_t. Taking expectation of (21) with respect to U_t gives

$$\overline{\mathcal{E}(S_t \to S_t \oplus U_t)} \leq 2n\nu\mathcal{E}_u = 2nh^{-1}(\alpha)\mathcal{E}_u. \qquad (22)$$

Note that as n approaches infinity, and because of Lemma 2.2, the entropy rate $\log(|\mathcal{Q}_*|)$ of the source U asymptotically equals αn. Thus, as $n \to \infty$, the utilization approaches α while the expected energy per bit remains less than or equal to

$$\frac{2nh^{-1}(\alpha)\mathcal{E}_u}{\alpha n} = \frac{2h^{-1}(\alpha)}{\alpha}\,\mathcal{E}_u. \qquad \square$$

For $\lambda = 0$ (the NSM case), the bound of Theorem 2.1 was also established in [13]. Our result though accounts for $\lambda > 0$ as well, that is, for the case of modern technologies where there is energy coupling between different lines of the bus, the DSM case.

III. Coding Theorems From Stationary Ergodic Processes

The simple bound that was established in the previous section assumed a basic differential encoding. In that case, the state of the bus at each time t depended only on the state at time $t-1$. This introduced the natural question: Is it possible to achieve more energy reduction using encoders with higher order memory? We answer the question affirmatively by presenting an example. This example also motivates a systematic study of the problem. Moreover, *the method developed here applies to other computational models as well.*

To proceed it is important to simplify the notation. From now on we will identify the vector elements

$$(0, 0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1)$$

of $\mathcal{Q}$ (and states of the bus) with the numbers $0, 1, \ldots, 2^n - 1$, respectively. In this notation, $\mathcal{E}(i \to j)$ is the energy required for the transition from state i to state j of the bus.

Theorem 3.1: Let $\mathcal{M}$ be a Markov source with the 2^n states, $0, 1, \ldots, 2^n - 1$. Let $\nu > 0$ and suppose that the probability of transition from state i to state j is given by

$$\Pr(j \,|\, i) = \frac{\exp(-\nu\mathcal{E}(i \to j))}{\sum_{k=0}^{2^n-1} \exp(-\nu\mathcal{E}(i \to k))}. \qquad (23)$$

Let $p(i)$, $i = 0, 1, \ldots, 2^n - 1$, denote the steady-state distribution of $\mathcal{M}$ and $\mathcal{H}(\mathcal{M})$ denote its entropy rate. Then the limiting expected energy per bit $\mathcal{E}_b$ at utilization $\mathcal{H}(\mathcal{M})/n$ of the bus satisfies the inequality

$$\mathcal{E}_b\left(\frac{\mathcal{H}(\mathcal{M})}{n}\right) \leq \frac{1}{\nu}\left[1 - \frac{\sum_{j=0}^{2^n-1} p(j)\log\left(\sum_{r=0}^{2^n-1}\exp(-\nu\mathcal{E}(j \to r))\right)}{\mathcal{H}(\mathcal{M})}\right]. \qquad (24)$$

Proof: First note that, by definition, the Markov process $\mathcal{M}$ is irreducible and aperiodic, therefore, the stationary distribution $p(i)$ exists and is unique. Also note that the transition energy, expression (1), is symmetric with respect to the starting and ending states, that is, $\mathcal{E}(i \to j) = \mathcal{E}(j \to i)$ for all i, j. This allows us to write $p(i)$ explicitly as (see, for example, [21])

$$p(i) = \frac{\sum_{k=0}^{2^n-1}\exp(-\nu\mathcal{E}(i \to k))}{\sum_{j=1}^{2^n-1}\sum_{k=0}^{2^n-1}\exp(-\nu\mathcal{E}(j \to k))}. \qquad (25)$$

In the cases where the energy cost function is not symmetric, $p(i)$ can be computed using standard methods. We endow the

process $\mathcal{M}$ with the steady-state distribution. The stationary Markov process $\mathcal{M}$ is irreducible, aperiodic, and therefore ergodic.

Let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_L)$ denote a sequence of L successive states of the Markov process $\mathcal{M}$. Distinct sequences of this type will be indexed by a bold-face superscript, $\underline{\sigma}^{\boldsymbol{i}} = (\sigma_1^{\boldsymbol{i}}, \ldots, \sigma_L^{\boldsymbol{i}})$. According to the Shannon–MacMillan–Breiman theorem [4], for every $\epsilon > 0$, there exists $L_0 > 0$ such that, for every $L \geq L_0$, there exists a set of sequences $\underline{\sigma}$ of length L, namely

$$\mathcal{T}(L) = \{\underline{\sigma}^{\boldsymbol{i}},\ \boldsymbol{i} = 1, 2, \ldots, |\mathcal{T}(L)|\} \tag{26}$$

such that

$$2^{L(\mathcal{H}(\mathcal{M})-\epsilon)} \leq |\mathcal{T}(L)| \leq 2^{L(\mathcal{H}(\mathcal{M})+\epsilon)} \tag{27}$$

and

$$\mathcal{H}(\mathcal{M}) - \epsilon \leq \frac{-\log \Pr(\sigma_1^{\boldsymbol{i}}, \ldots, \sigma_L^{\boldsymbol{i}})}{L} \leq \mathcal{H}(\mathcal{M}) + \epsilon \tag{28}$$

for every $\boldsymbol{i} = 1, 2, \ldots, |\mathcal{T}(L)|$. The sequences which belong to $\mathcal{T}(L)$ are referred to as "ϵ-typical." We take the set of typical sequences $\mathcal{T}(L)$ of the Markov process $\mathcal{M}$ as our bus code, that is, $\mathcal{C}(L) = \mathcal{T}(L)$, and we choose the elements of $\mathcal{C}(L)$ with equal probability. Then the utilization α of the bus, using code $\mathcal{C}(L)$, is such that

$$\frac{\mathcal{H}(\mathcal{M})}{n} - \frac{\epsilon}{n} \leq \alpha \leq \frac{\mathcal{H}(\mathcal{M})}{n} + \frac{\epsilon}{n}. \tag{29}$$

Now, writing

$$\Pr(\sigma_1^{\boldsymbol{i}}, \ldots, \sigma_L^{\boldsymbol{i}}) = p(\sigma_1^{\boldsymbol{i}}) \prod_{k=2}^{L} \Pr(\sigma_k^{\boldsymbol{i}} \,|\, \sigma_{k-1}^{\boldsymbol{i}})$$

and replacing it into (28) we get

$$\begin{aligned}\mathcal{H}(\mathcal{M}) - \epsilon + \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L} &\leq -\frac{1}{L}\sum_{k=2}^{L} \log \Pr(\sigma_k^{\boldsymbol{i}} \,|\, \sigma_{k-1}^{\boldsymbol{i}}) \\ &\leq \mathcal{H}(\mathcal{M}) + \epsilon + \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L}. \end{aligned} \tag{30}$$

Taking account of (23) into the above inequalities we get

$$\begin{aligned}&\mathcal{H}(\mathcal{M}) - \epsilon + \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L} \\ &\quad\leq \frac{1}{L}\sum_{k=2}^{L} \nu \mathcal{E}(\sigma_{k-1}^{\boldsymbol{i}} \to \sigma_k^{\boldsymbol{i}}) \\ &\qquad + \sum_{j=0}^{2^n-1} \tilde{p}^{\boldsymbol{i}}(j) \log\left(\sum_{k=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to k))\right) \\ &\quad\leq \mathcal{H}(\mathcal{M}) + \epsilon + \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L} \end{aligned} \tag{31}$$

where we have set

$$\tilde{p}^{\boldsymbol{i}}(j) = \frac{N^{\boldsymbol{i}}(j)}{L}$$

and $N^{\boldsymbol{i}}(j)$ is the number of occurrences of state j in the sequence $\sigma_2^{\boldsymbol{i}}, \sigma_3^{\boldsymbol{i}}, \ldots, \sigma_L^{\boldsymbol{i}}$. By summing up the above inequalities over $\boldsymbol{i} = 1, 2, \ldots, |\mathcal{T}(L)|$ and dividing the sum by $|\mathcal{T}(L)|$ we obtain

$$\begin{aligned}&\mathcal{H}(\mathcal{M}) - \epsilon + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L} \\ &\leq \nu \frac{1}{L}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{1}{|\mathcal{T}(L)|}\left(\sum_{k=2}^{L} \mathcal{E}(\sigma_{k-1}^{\boldsymbol{i}} \to \sigma_k^{\boldsymbol{i}})\right) \\ &\quad + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|}\sum_{j=0}^{2^n-1} \tilde{p}^{\boldsymbol{i}}(j) \log\left(\sum_{k=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to k))\right) \\ &\leq \mathcal{H}(\mathcal{M}) + \epsilon + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L}. \end{aligned}$$

Using (6), the first summand in the middle term of the preceding inequality can be written as

$$\frac{1}{L}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{1}{|\mathcal{T}(L)|}\left(\sum_{k=2}^{L} \mathcal{E}(\sigma_{k-1}^{\boldsymbol{i}} \to \sigma_k^{\boldsymbol{i}})\right) = \mathcal{E}_{\text{av}}(\mathcal{C}(L)) - \frac{e(\mathcal{C})}{L}$$

where $0 \leq e(\mathcal{C}) \leq \mathcal{E}_{\max}$. Therefore, we have

$$\begin{aligned}&\mathcal{H}(\mathcal{M}) - \epsilon + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L} \\ &\leq \nu \mathcal{E}_{\text{av}}(\mathcal{C}(L)) - \nu \frac{e(\mathcal{C})}{L} \\ &\quad + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|}\sum_{j=0}^{2^n-1} \tilde{p}^{\boldsymbol{i}}(j) \log\left(\sum_{k=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to k))\right) \\ &\leq \mathcal{H}(\mathcal{M}) + \epsilon + \frac{1}{|\mathcal{T}(L)|}\sum_{\boldsymbol{i}=1}^{|\mathcal{T}(L)|} \frac{\log p(\sigma_1^{\boldsymbol{i}})}{L}. \end{aligned}$$

The above result holds for every $L \geq L_0$. Also, $\mathcal{E}(\mathcal{C}(L)) \leq \mathcal{E}_{\max} < \infty$. Letting $L \to \infty$ and using strong typicality [4], we have $\tilde{p}^{\boldsymbol{i}}(j) \to p(j)$ with probability one. Moreover, $\nu \frac{e(\mathcal{C})}{L} \to 0$. Thus, for sufficiently large L, we have

$$\begin{aligned}\mathcal{H}(\mathcal{M}) - 2\epsilon \leq\ &\nu \mathcal{E}_{\text{av}}(\mathcal{C}(L)) \\ &+ \sum_{j=0}^{2^n-1} p(j) \log\left(\sum_{k=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to k))\right) \\ \leq\ &\mathcal{H}(\mathcal{M}) + 2\epsilon \end{aligned} \tag{32}$$

with probability one. Since ϵ is arbitrary, an expected energy consumption arbitrarily close to

$$\frac{1}{\nu}\left[\mathcal{H}(\mathcal{M}) - \sum_{j=0}^{2^n-1} p(j) \log\left(\sum_{k=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to k))\right)\right]$$

is achievable at utilizations arbitrarily close to $\alpha = \mathcal{H}(\mathcal{M})/n$. Thus,

$$\left(\frac{\mathcal{H}(\mathcal{M})}{n}, \frac{1}{\nu}\left[1 - \frac{\sum_{j=0}^{2^n-1} p(j) \log\left(\sum_{r=0}^{2^n-1} \exp(-\nu \mathcal{E}(j \to r))\right)}{\mathcal{H}(\mathcal{M})}\right]\right)$$

is an achievable pair for every $\nu \geq 0$. □

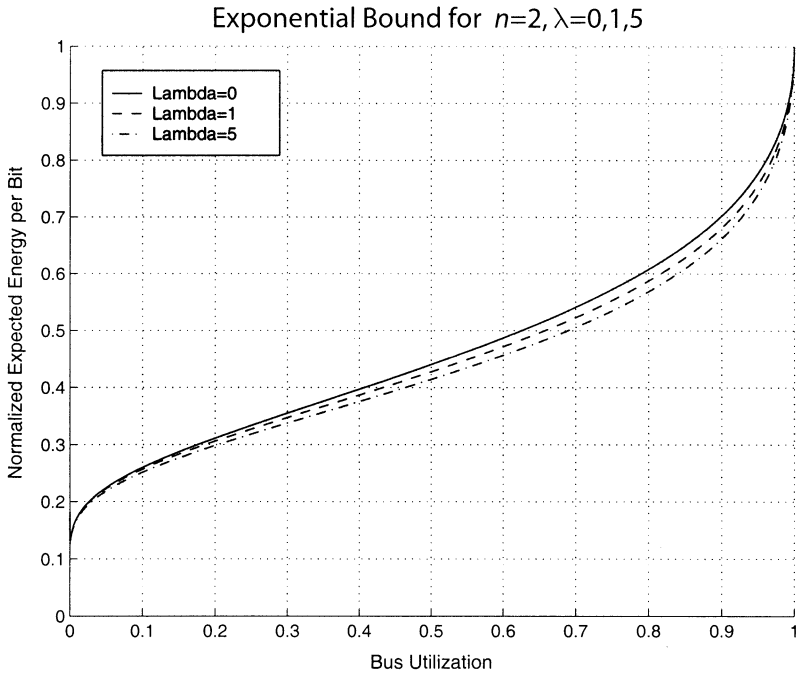

Fig. 2. The exponential bound for $n = 2$ and $\lambda = 0,\ 1,\ 5$. For $\lambda = 0$, the exponential and entropy bounds coincide.

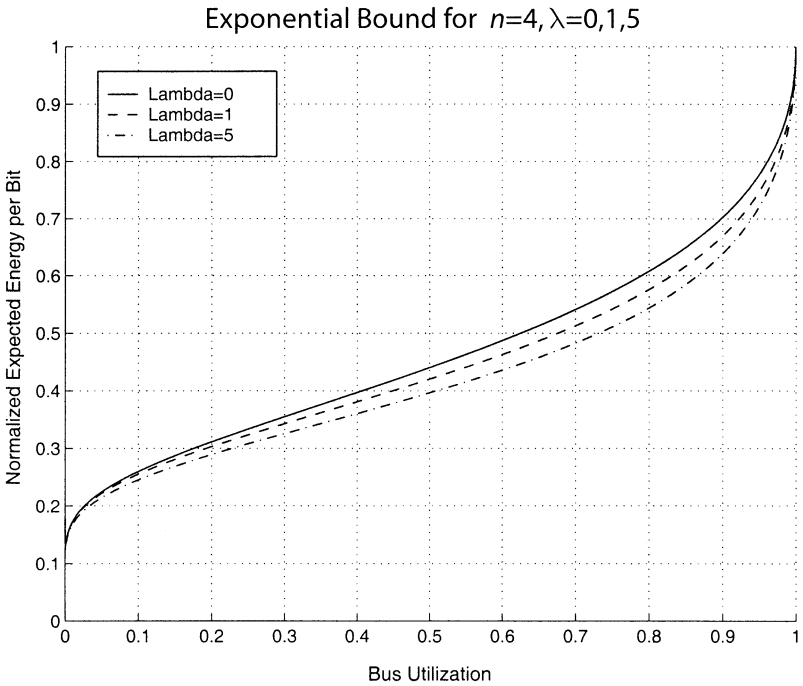

Fig. 3. The exponential bound for $n = 4$ and $\lambda = 0,\ 1,\ 5$. For $\lambda = 0$ the exponential and entropy bounds coincide.

The bound of Theorem 3.1 is referred to as the *exponential bound*. We have plotted the results of Theorem 3.1 in Figs. 2–4 for small values of n and with the vertical axis labeled by $\mathcal{E}_b(\alpha)/\mathcal{E}_u$. This bound is hard to compute for higher values of n since the Markov chain $\mathcal{M}$ has too many states. This makes the computation of the steady-state distribution numerically difficult for $n > 8$. For comparison, we have also plotted the bound of Theorem 2.1. We see in Figs. 2–4 that the bound of Theorem 3.1 is always better than the bound of Theorem 2.1 for all the plotted values. Moreover, it is seen in the figures, and it is easy to prove, that for $\lambda = 0$, the exponential bound and the entropy bound coincide.

Motivated by the above result, it is natural to ask if we can use other stationary ergodic processes, invoke again the Shannon–McMillan–Breiman theorem, and obtain stronger results. We proceed with the following construction:

Construction I: Let $\mathcal{X}$ denote a stationary ergodic stochastic process whose outcomes are elements of $\mathcal{Q} = \{0, 1\}^n$. We consider the set $\mathcal{X}_L$ consisting of all sequences $(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ of L successive outcomes of the

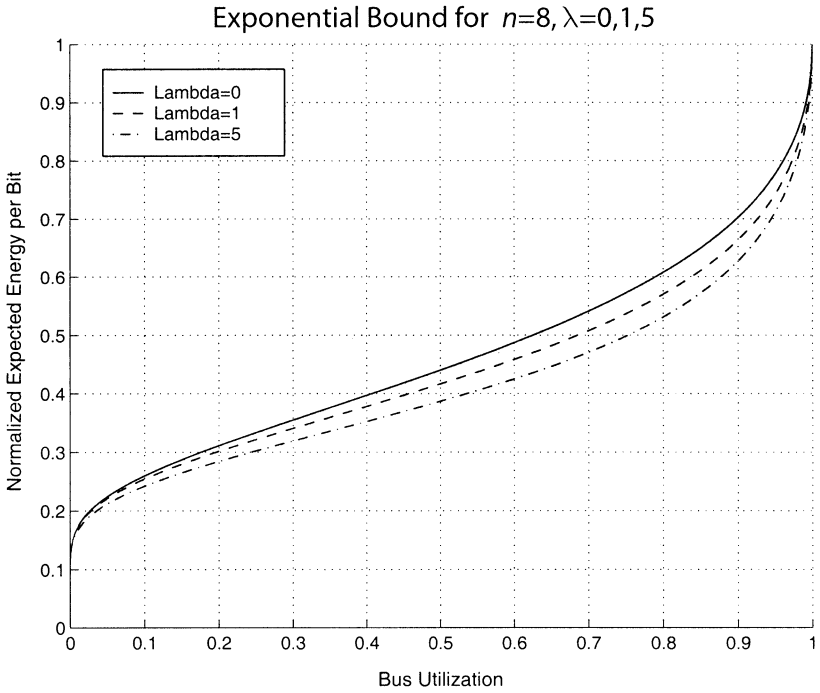

Fig. 4. The exponential bound for $n = 8$ and $\lambda = 0,\ 1,\ 5$. For $\lambda = 0$ the exponential and entropy bounds coincide.

stationary ergodic stochastic process $\mathcal{X}$. We have, of course, $\mathcal{X}_L \subset \mathcal{Q}^L$. Given $\epsilon > 0$ we also consider the subset $\mathcal{T}_L(\mathcal{X})$ of $\mathcal{X}_L$ that contains all the ϵ-typical sequences, that is, all $(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ such that

$$\mathcal{H}(\mathcal{X}) - \epsilon \leq -\frac{1}{L}\log \Pr(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L) \leq \mathcal{H}(\mathcal{X}) + \epsilon.$$

These two sets of sequences define, respectively, two codes $\mathcal{X}(L) = \mathcal{X}_L$ and $\mathcal{C}(L) = \mathcal{T}_L(\mathcal{X})$ of length L. A sequence $(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ in $\mathcal{Q}^L$ is chosen with probability $\Pr(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ if it is regarded as a codeword of $\mathcal{X}_L$. If the same sequence is regarded as a codeword of $\mathcal{C}(L)$, then it is chosen with probability $c(L)\Pr(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ or 0, where

$$c(L) = 1/\sum_{\sigma \in \mathcal{C}(L)} \Pr(\sigma)$$

according to whether $(\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L)$ is, or not, an ϵ-typical sequence.

As $L \to \infty$, we know from the Shannon–McMillan–Breiman theorem that $\Pr(\mathcal{T}_L(\mathcal{X})) \to 1$ and $c(L) \to 1$. This implies the following lemma.

Lemma 3.1: Let a stationary ergodic process $\mathcal{X}$ be given. As $L \to \infty$, the expected energy per bit $\mathcal{E}_b(\mathcal{C}(L))$ of the code $\mathcal{C}(L)$ consisting of the ϵ-typical sequences $\mathcal{T}_L(\mathcal{X})$ becomes equal to the expected energy per bit $\mathcal{E}_b(\mathcal{X}(L))$ of the code $\mathcal{X}(L)$. The same is true for the utilizations of the two codes.

Proof: Let $\mathcal{E}_{\max} = \max_{\sigma,\, \omega \in \mathbb{Z}_2^n} \mathcal{E}(\sigma \to \omega)$. As $L \to \infty$, the set of all the codewords in $\mathcal{X}(L)$ that are not in $\mathcal{C}(L)$ has an arbitrary small probability δ. This means that they contribute at most $\delta\mathcal{E}_{\max}$ to the expected energy per bus transition. This contribution can be made as small as desired. Since $c(L) \to 1$ as $L \to \infty$, the utilizations afforded by $\mathcal{C}(L)$ and $\mathcal{X}(L)$ get arbitrary close to each other as $L \to \infty$. Combining these two observations yields the result. □

Definition 3.1: Let $\mathcal{X}$ be a stationary (not necessarily ergodic) process in $\mathcal{Q}$. *The utilization*, the *expected energy consumption* and the *expected energy consumption per bit* of process $\mathcal{X}$ are defined, respectively, as

$$\alpha = \mathcal{H}(\mathcal{X})/n, \tag{33}$$

$$\mathcal{E}_{\text{av}}(\mathcal{X}) = \sum_{\sigma \in \mathcal{Q}} \sum_{\omega \in \mathcal{Q}} \Pr(X_1 = \sigma,\, X_2 = \omega)\mathcal{E}(\sigma \to \omega) = \overline{\mathcal{E}(\sigma \to \omega)} \tag{34}$$

and

$$\mathcal{E}_b(\mathcal{X}) = \frac{\mathcal{E}_{\text{av}}(\mathcal{X})}{\mathcal{H}(\mathcal{X})}. \tag{35}$$

The following two definitions are analogous to that based on sequences of codes.

Definition 3.2: We define a pair (α, β), where α denotes the utilization of the bus and β the expected energy per bit, to be *achievable* by a class $\boldsymbol{X}$ of stationary processes in $\mathcal{Q}$, if and only if there exists an infinite sequence of processes $\mathcal{X}_i$ in $\boldsymbol{X}$ (not necessarily distinct) such that $\mathcal{H}(\mathcal{X}_i) \to \alpha_1 n$, for some $\alpha_1 \geq \alpha$ and $\mathcal{E}_b(\mathcal{X}_i) \to \beta_1$, for some $\beta_1 \leq \beta$, as $i \to \infty$.

Definition 3.3: We define the *limiting expected energy consumption per bit* $\mathcal{E}_b$ at utilization $\alpha \in [0, 1]$ of a class $\boldsymbol{X}$ of stationary processes to be the function

$$\mathcal{E}_b(\alpha, \boldsymbol{X}) = \inf\{\beta \,|\, \text{The pair } (\alpha, \beta) \text{ is achievable by the class of processes } \boldsymbol{X}\}. \tag{36}$$

We are now in the position to prove the following important theorem.

Theorem 3.2: Let $\mathcal{X}$ be a stationary (not necessarily ergodic) process in $\mathcal{Q}$ of utilization α and expected energy per bit $\mathcal{E}_b(\mathcal{X})$. The pair $(\alpha, \mathcal{E}_b(\mathcal{X}))$ is acheivable[1] by a sequence of codes whose codewords are typical (finite) sequences of stationary ergodic Markov processes.

Proof: Let $\mathcal{X}$ be a stationary stochastic process with outcomes $X_1, X_2, X_3, \ldots$ at time $t = 1, 2, 3, \ldots$, respectively, and with probability distribution $\Pr(X_1, X_2, \ldots, X_k)$ for $k = 1, 2, \ldots$. We can assume that $\Pr(X_1 = \sigma) > 0$ for every state of $\mathcal{X}$, otherwise, we can "remove" σ from the set of states of $\mathcal{X}$. Let $\mathcal{M}$ be the stationary Markov process with stationary distribution $p(\sigma) = \Pr(X_1 = \sigma)$ and transition probabilities $\Pr(\omega \,|\, \sigma) = \Pr(X_2 = \omega \,|\, X_1 = \sigma)$. We first observe that

$$\begin{aligned}\mathcal{H}(\mathcal{X}) &= \lim_{k\to\infty} \mathcal{H}(X_k \,|\, X_{k-1}, X_{k-2}, \ldots, X_1) \\ &\le \mathcal{H}(X_2 \,|\, X_1) = \mathcal{H}(\mathcal{M}). \end{aligned} \tag{37}$$

Suppose for the moment that $\mathcal{M}$ is an ergodic process. Let $\epsilon > 0$ and $\mathcal{M}(L)$ be the set of all ϵ-typical sequences of length L of the process $\mathcal{M}$. We can conclude from an application of the Shannon–McMillan–Breiman theorem that $\mathcal{M}(L)$ has at least $2^{L(\mathcal{H}(\mathcal{M})-\epsilon)}$ codewords (for L sufficiently large). Therefore, choosing ϵ sufficiently small, the utilization of the bus by the code $\mathcal{M}(L)$ can be as close to $\mathcal{H}(\mathcal{M})/n$ as desired. Note that $\mathcal{H}(\mathcal{M})/n \ge \mathcal{H}(\mathcal{X})/n$. Moreover, the ergodicity of $\mathcal{M}$ along with Lemma 3.1 imply that the expected energy consumption $\mathcal{E}_{\text{av}}(\mathcal{M}(L))$ of the code $\mathcal{M}(L)$ approaches the expected energy consumption $\mathcal{E}_{\text{av}}(\mathcal{M})$ of the process $\mathcal{M}$ as $L \to \infty$. By the definition of process $\mathcal{M}$ it is also true that $\mathcal{E}_{\text{av}}(\mathcal{M}) = \mathcal{E}_{\text{av}}(\mathcal{X})$ where

$$\mathcal{E}_{\text{av}}(\mathcal{X}) = \sum_{\sigma\in\mathcal{Q}} \sum_{\omega\in\mathcal{Q}} \Pr(X(1) = \sigma, X(2) = \omega)\mathcal{E}(\sigma \to \omega).$$

Thus, using typical sequences of the ergodic Markov process $\mathcal{M}$ we can construct a family of codes $\mathcal{M}(L)$ such that

$$\liminf_{L\to\infty} \mathcal{H}(\mathcal{M}(L)) \ge \mathcal{H}(\mathcal{X}) \text{ and } \lim_{L\to\infty} \mathcal{E}_{\text{av}}(\mathcal{M}(L)) = \mathcal{E}_{\text{av}}(\mathcal{X}).$$

This concludes the theorem in the case that process $\mathcal{M}$ is ergodic.

It remains to treat the case where the process $\mathcal{M}$ is not ergodic. Consider the matrix $\boldsymbol{P}$ with the i, jth element

$$\boldsymbol{P}_{ij} = \Pr(X(2) = j \,|\, X(1) = i)$$

that is, the transition probability matrix of the Markov process $\mathcal{M}$. Then we claim that some of the elements of $\boldsymbol{P}$ are zeroes. If not, $\boldsymbol{P}$ is an irreducible matrix and $\mathcal{M}$ is an aperiodic irreducible Markov chain and hence is ergodic. Consider the row matrix $\boldsymbol{q}$ whose ith element is $\Pr(X(1) = i)$. Then, $\boldsymbol{qP} = \boldsymbol{P}$ and all the elements of $\boldsymbol{q}$ are positive. (This is true because we have assumed that process $\mathcal{X}$ visits all its states and so process $\mathcal{M}$ is irreducible.) For every $0 \le \delta \le 1$, we set $\boldsymbol{P}_\delta = (1 - \delta)\boldsymbol{P} + \delta \mathbf{1}^T \boldsymbol{q}$, where $\mathbf{1}$ is a row matrix with all of its elements being 1. We also have $\boldsymbol{qP}_\delta = \boldsymbol{q}$ for every $\delta \ge 0$.

We now consider the Markov source $\mathcal{M}_\delta$ whose stationary distribution is $\boldsymbol{q}$ and whose state transition matrix is $\boldsymbol{P}_\delta$. For every $\delta > 0$, the matrix $\boldsymbol{P}_\delta$ has only positive elements and thus, $\mathcal{M}_\delta$ is a stationary irreducible ergodic Markov process. By the continuity of the entropy, the entropy of $\mathcal{M}_\delta$ can be made arbitrary close to the entropy of $\mathcal{M}$. Because $\boldsymbol{P}_\delta$ can get as close to $\boldsymbol{P}$ as desired, the expected energy consumption by typical sequences of $\mathcal{M}_\delta$ approaches that of $\mathcal{X}$. Thus, as $\delta \to 0$, we can get arbitrary close to a superior or at least equivalent tradeoff between energy and utilization than that of the stationary process $\mathcal{X}$. □

[1]According to Definition 2.6.

The preceding result motivates us to ask the following question: Given a (utilization, expected energy per bit) pair (α, β), achievable by a sequence of codes (Definition 2.6), is it possible to construct a (possibly different) sequence of codes, consisting of typical sequences of ergodic Markov sources, that achieve (α, β)?

Our goal is to answer this question affirmatively. We start by introducing the following construction that is in some way the inverse of that used before. Here, we start with a given code and construct a stationary process of at least as high utilization and the same expected energy consumption as that of the code.

Construction II: Let $\mathcal{C}$ denote a code of length L and utilization α. We will construct a stationary stochastic process $\mathcal{X}(\mathcal{C})$ from $\mathcal{C}$. To do so, we will first construct an interim stochastic process $\mathcal{Y}$: $Y_1, Y_2, Y_3, \ldots$ by describing the joint distribution of the indexed random variables $Y_1, Y_2, Y_3, \ldots$.

Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3, \ldots$ denote the random vectors of length L with

$$\boldsymbol{Y}_k = (Y_{(k-1)L+1}, Y_{(k-1)L+2}, \ldots, Y_{kL}), \quad \text{for } k = 1, 2, \ldots.$$

We define the random vectors $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3, \ldots$ to be mutually independent and such that for every $\boldsymbol{c} \in \mathcal{Q}^L$ we have

$$\Pr(\boldsymbol{Y}_k = \boldsymbol{c}) = \begin{cases} \Pr(\boldsymbol{c}), & \text{if } \boldsymbol{c} \in \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$

These define the interim stochastic process $\mathcal{Y}$ completely. Now for $l = 1, 2, \ldots$ we define a sequence of distributions (on the random variables $X_1, X_2, \ldots$) by letting

$$\begin{aligned}&\Pr\left((X_1 = x_1, X_2 = x_2, \ldots, X_l = x_l)\right) \\ &\quad = \frac{1}{L}\sum_{i=0}^{L-1} \Pr\left((Y_{1+i} = x_1, Y_{2+i} = x_2, \ldots, Y_{l+i} = x_l)\right).\end{aligned}$$

Note that the right-hand side of the expression is a shift-and-average operation. This sequence of distributions is *consistent* in the sense that

$$\begin{aligned}&\Pr\left((X_1 = x_1, X_2 = x_2, X_3 = x_3, \ldots, X_{l-1} = x_{l-1})\right) \\ &\quad = \sum_{x_l\in\mathcal{Q}} \Pr\left((X_1 = x_1, X_2 = x_2, X_3 = x_3, \ldots, X_l = x_l)\right).\end{aligned}$$

A theorem of Kolmogorov [21, p. 21] implies the existence of a random process $\mathcal{X}(\mathcal{C})$: $X_1, X_2, X_3, \ldots$ characterized by the family of the above distributions. Next, we prove that the process $\mathcal{X}(\mathcal{C})$ is stationary. To do so, we first observe that the process $\mathcal{Y}$ is cyclo-stationary with period L in the sense that

$$\begin{aligned}&\Pr\left((Y_1, Y_2, Y_3, \ldots, Y_l)\right) \\ &\qquad = \Pr\left((Y_{1+mL}, Y_{2+mL}, Y_{3+mL}, \ldots, Y_{l+mL})\right.\end{aligned}$$

for all $l=1, 2, \ldots$ and $m=0, 1, \ldots$. If $L=1$ then $\boldsymbol{Y}_k=(Y_k)$ for all $k = 1, 2, \ldots$ and so the processes $\mathcal{Y}$ and $\mathcal{X}$ are both stationary. Suppose now that $L>1$ and $j\geq 1$, $l\geq 1$ are given. Then

$$\begin{aligned}
&\Pr(X_{1+j}=y_1, \ldots, X_{l+j}=y_l)\\
&\quad=\sum_{x_1, x_2, \ldots, x_j} \Pr(X_1=x_1, \ldots, X_j=x_j,\\
&\qquad X_{1+j}=y_1, \ldots, X_{l+j}=y_l)\\
&\quad=\sum_{x_1, x_2, \ldots, x_j} \frac{1}{L}\sum_{i=0}^{L-1} \Pr(Y_{1+i}=x_1, \ldots, Y_{j+i}=x_j,\\
&\qquad Y_{1+j+i}=y_1, Y_{2+j+i}=y_2, \ldots, Y_{l+j+i}=y_l)\\
&\quad=\frac{1}{L}\sum_{i=0}^{L-1}\sum_{x_1, x_2, \ldots, x_j} \Pr(Y_{1+i}=x_1, \ldots, Y_{j+i}=x_j,\\
&\qquad Y_{1+j+i}=y_1, Y_{2+j+i}=y_2, \ldots, Y_{l+j+i}=y_l)\\
&\quad=\frac{1}{L}\sum_{i=0}^{L-1} \Pr(Y_{1+j+i}=y_1, Y_{2+j+i}=y_2, \ldots, Y_{l+j+i}=y_l).
\end{aligned}$$

As i runs in the set $0, 1, 2\ldots, L-1$, the value $i+j \bmod (L)$ also runs in the same set. Combining this observation and the cyclo-stationarity of the process $\mathcal{Y}$, we conclude that

$$\begin{aligned}
&\frac{1}{L}\sum_{i=0}^{L-1} \Pr\left(Y_{1+j+i}=y_1, Y_{2+j+i}=y_2, \ldots, Y_{l+j+i}=y_l\right)\\
&\quad=\frac{1}{L}\sum_{i=0}^{L-1} \Pr\left(Y_{1+i}=y_1, \ldots, Y_{l+i}=y_l\right)\\
&\quad=\Pr\left(X_1=y_1, X_2=y_2, X_3=y_3, \ldots, X_l=y_l\right).
\end{aligned}$$

Thus, the process $\mathcal{X}(\mathcal{C})$ is stationary.

Lemma 3.2: The entropy rate $\mathcal{H}(\mathcal{X}(\mathcal{C}))$ of the random process $\mathcal{X}(\mathcal{C})$ is greater than or equal to the entropy

$$\mathcal{H}(\mathcal{C})=-\frac{1}{L}\sum_{\boldsymbol{c}\in\mathcal{C}} \Pr(\boldsymbol{c})\log\Pr(\boldsymbol{c})$$

of the code $\mathcal{C}$.

Proof: By the definition of the process $\mathcal{X}(\mathcal{C})$ we have

$$\Pr(X_1, X_2, \ldots, X_l)=\sum_{i=0}^{L-1}\frac{1}{L}\Pr(Y_{1+i}, Y_{2+i}, \ldots, Y_{l+i}).$$

Since the entropy function is concave, for every $l = 1, 2, \ldots$, we also have

$$\mathcal{H}(X_1, X_2, \ldots, X_l)\geq\frac{1}{L}\sum_{i=0}^{L-1}\mathcal{H}(Y_{1+i}, Y_{2+i}, \ldots, Y_{l+i}).$$

Dividing by l and letting $l\to\infty$ we obtain $\mathcal{H}(\mathcal{X})\geq\mathcal{H}(\mathcal{Y})$. But

$$\begin{aligned}
\mathcal{H}(\mathcal{Y})&=\lim_{k\to\infty}\frac{\mathcal{H}(Y_1, Y_2, \ldots, Y_k)}{k}\\
&=\lim_{k\to\infty}\frac{\mathcal{H}(Y_1, Y_2, \ldots, Y_{Lk})}{kL}\\
&=\lim_{k\to\infty}\sum_{r=1}^{k}\frac{\mathcal{H}(Y_{(r-1)L+1}, Y_{(r-1)L+2}, \ldots, Y_{rL})}{kL}\\
&=\frac{1}{L}H(Y_1, Y_2\ldots, Y_L)
\end{aligned}$$

where we have used the cyclo-stationarity of $\mathcal{Y}$ and the independence of the random vectors $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Y}_3, \ldots$. By Definition 2.2, we have

$$\frac{1}{L}H(Y_1, \ldots, Y_L)=\mathcal{H}(\mathcal{C}).$$

This concludes the proof. □

Lemma 3.3: The expected energy consumption $\mathcal{E}_{\text{av}}(\mathcal{X})$ of the process $\mathcal{X}(\mathcal{C})$ equals the expected energy consumption $\mathcal{E}_{\text{av}}(\mathcal{C})$ of the code $\mathcal{C}$.

Proof: The energy consumption of a sequence $\sigma_1, \sigma_2, \ldots, \sigma_l$ of elements in $\mathcal{Q}$ is (Definition 3.1)

$$\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)=\sum_{i=1}^{l-1}\mathcal{E}(\sigma_i\to\sigma_{i+1}).$$

The expected energy consumption of the stationary process $\mathcal{X}(\mathcal{C})$ is given by

$$\begin{aligned}
\mathcal{E}_{\text{av}}(\mathcal{X})&=\overline{\mathcal{E}(\sigma\to\omega)}\\
&=\lim_{l\to\infty}\frac{\overline{\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)}}{l-1},
\end{aligned}$$

where the expectation $\overline{\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)}$ is taken with respect to the probability distribution $\Pr(X_1=\sigma_1, \ldots, X_l=\sigma_l)$. However,

$$\Pr(X_1, X_2, \ldots, X_l)=\sum_{i=0}^{L-1}\frac{1}{L}\Pr(Y_{1+i}, Y_{2+i}, \ldots, Y_{l+i})$$

thus,

$$\begin{aligned}
\overline{\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)}&=\frac{1}{L}\sum_{i=0}^{L-1}\sum_{\sigma_1, \ldots, \sigma_l}\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)\\
&\qquad\cdot\Pr(Y_{1+i}=\sigma_1, \ldots, Y_{l+i}=\sigma_l).
\end{aligned}$$

Now, for $i = 0, 1, 2, \ldots, L-1$, we set $l_i = L-i+1$ and $l^i = L\lfloor\frac{l}{L}\rfloor - i$. For every l sufficiently large (e.g., greater than $3L$) we have

$$\begin{aligned}
\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)&=\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_{l_i})\\
&\quad+\mathcal{E}(\sigma_{l_i}, \sigma_{l_i+1}, \ldots, \sigma_{l^i})+\mathcal{E}(\sigma_{l^i}, \sigma_{l^i+1}, \ldots, \sigma_l)
\end{aligned}$$

where we agree that the expected energy of any trivial 1-element sequence is zero, i.e., $\mathcal{E}(\sigma)=0$ for every $\sigma\in\mathcal{Q}$. Then we have

$$\begin{aligned}
\mathcal{E}(\sigma_{l_i}, \sigma_{l_i+1}, \ldots\sigma_{l^i})&\leq\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)\\
&\leq\mathcal{E}(\sigma_{l_i}, \sigma_{l_i+1}, \ldots, \sigma_{l^i})+3L\mathcal{E}_{\max}
\end{aligned}\quad(38)$$

were $\mathcal{E}_{\max}=\max_{\sigma,\omega\in\mathcal{Q}}\mathcal{E}(\sigma\to\omega)$. We conclude that

$$\begin{aligned}
&\overline{\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)}\\
&\quad=\frac{1}{L}\sum_{i=0}^{L-1}\sum_{\sigma_1, \ldots, \sigma_l}\mathcal{E}(\sigma_1, \sigma_2, \ldots, \sigma_l)\\
&\qquad\cdot\Pr(Y_{1+i}=\sigma_1, \ldots, Y_{l+i}=\sigma_l)\\
&\quad\leq\frac{1}{L}\sum_{i=0}^{L-1}\sum_{\sigma_1, \ldots, \sigma_l}\left(\mathcal{E}(\sigma_{l_i}, \ldots, \sigma_{l^i})+3L\mathcal{E}_{\max}\right)\\
&\qquad\cdot\Pr(Y_{1+i}=\sigma_1, \ldots, Y_{l+i}=\sigma_l)\\
&\quad\leq\left[\frac{1}{L}\sum_{i=0}^{L-1}\sum_{\sigma_{l_i}, \ldots, \sigma_{l^i}}\mathcal{E}(\sigma_{l_i}, \ldots, \sigma_{l^i})\right.\\
&\qquad\left.\cdot\Pr(Y_{l_i+i}=\sigma_{l_i}, \ldots, Y_{l^i+i}=\sigma_{l^i})\right]+3L\mathcal{E}_{\max}.
\end{aligned}\quad(39)$$

However,

$$\begin{aligned}&\Pr(Y_{l_i+i}=\sigma_{l_i},\ldots,Y_{l^i+i}=\sigma_{l^i})\\&\quad=\Pr\left(Y_{L+1}=\sigma_{l_i},\ldots,Y_{L\lfloor\frac{l}{L}\rfloor}=\sigma_{l^i}\right)\\&\quad=\Pr\left(\left(\boldsymbol{Y}_2,\boldsymbol{Y}_3,\ldots,\boldsymbol{Y}_{\lfloor\frac{l}{L}\rfloor}\right)=(\sigma_{l_i},\ldots,\sigma_{l^i})\right).\end{aligned}$$

Since the random vectors $\boldsymbol{Y}_2,\boldsymbol{Y}_3,\ldots,\boldsymbol{Y}_{\lfloor\frac{l}{L}\rfloor}$ are independent, we can write

$$\begin{aligned}&\Pr(Y_{l_i+i}=\sigma_{l_i},\ldots,Y_{l^i+i}=\sigma_{l^i})\\&\quad=\Pr(\boldsymbol{Y}_2=(\sigma_{l_i},\ldots,\sigma_{l_i+L-1}))\\&\qquad\cdot\Pr(\boldsymbol{Y}_3=(\sigma_{l_i+L},\ldots,\sigma_{l_i+2L-1}))\\&\qquad\cdots\Pr\left(\boldsymbol{Y}_{\lfloor\frac{l}{L}\rfloor}=(\sigma_{l^i-L+1},\ldots,\sigma_{l^i})\right).\end{aligned}$$

In addition, we can write

$$\begin{aligned}&\mathcal{E}(\sigma_{l_i},\ldots,\sigma_{l^i})\\&\quad=\mathcal{E}(\sigma_{l_i},\ldots,\sigma_{l_i+L-1})+\mathcal{E}(\sigma_{l_i+L-1}\to\sigma_{l_i+L})\\&\qquad+\mathcal{E}(\sigma_{l_i+L},\ldots,\sigma_{l_i+2L-1})+\mathcal{E}(\sigma_{l_i+2L-1}\to\sigma_{l_i+2L})\\&\qquad+\cdots+\mathcal{E}(\sigma_{l^i-L+1},\ldots,\sigma_{l^i}).\end{aligned}$$

The expected energy of the code $\mathcal{C}$ is by its Definition 2.3

$$\begin{aligned}\mathcal{E}_{\rm av}(\mathcal{C})=&\frac{1}{L}\sum_{\sigma_{l_i},\ldots,\sigma_{l_i+L-1}}\mathcal{E}(\sigma_{l_i},\ldots,\sigma_{l_i+L-1})\\&\cdot\Pr(\boldsymbol{Y}_2=(\sigma_{l_i},\ldots,\sigma_{l_i+L-1}))\\&+\frac{1}{L}\sum_{\sigma_{l_i},\ldots,\sigma_{l_i+2L-1}}\Pr(\boldsymbol{Y}_2=(\sigma_{l_i},\ldots,\sigma_{l_i+L-1}))\\&\cdot\Pr(\boldsymbol{Y}_3=(\sigma_{l_i+L},\ldots,\sigma_{l_i+2L-1}))\\&\cdot\mathcal{E}(\sigma_{l_i+L-1}\to\sigma_{l_i+L}).\end{aligned}$$

Using the cyclo-stationarity of $\mathcal{Y}$ and the expressions above, we arrive at

$$\begin{aligned}&\sum_{\sigma_{l_i},\ldots,\sigma_{l^i}}\mathcal{E}(\sigma_{l_i},\ldots,\sigma_{l^i})\Pr(Y_{l_i+i}=\sigma_{l_i},\ldots,Y_{l^i+i}=\sigma_{l^i})\\&\qquad=L\left(\left\lfloor\frac{l}{L}\right\rfloor-1\right)\mathcal{E}_{\rm av}(\mathcal{C})-\frac{e(\mathcal{C})}{L}\end{aligned}$$

where $e(\mathcal{C})$ is as in (6). Finally, replacing the above into (39), we get

$$\overline{\mathcal{E}(\sigma_1,\sigma_2,\ldots,\sigma_l)}\le l\cdot\mathcal{E}_{\rm av}(\mathcal{C})+3L\mathcal{E}_{\max}.\tag{40}$$

Similarly, we can prove the inequality

$$L\left(\left\lfloor\frac{l}{L}\right\rfloor-1\right)\mathcal{E}_{\rm av}(\mathcal{C})-\mathcal{E}_{\max}\le\overline{\mathcal{E}(\sigma_1,\sigma_2,\ldots,\sigma_l)}.\tag{41}$$

Combining (40) and (41) and dividing by l we have

$$\begin{aligned}\frac{L}{l}\left(\left\lfloor\frac{l}{L}\right\rfloor-1\right)\mathcal{E}_{\rm av}(\mathcal{C})-\frac{\mathcal{E}_{\max}}{l}&\le\frac{\overline{\mathcal{E}(\sigma_1,\sigma_2,\ldots,\sigma_l)}}{l}\\&\le\mathcal{E}_{\rm av}(\mathcal{C})+\frac{3L\mathcal{E}_{\max}}{l}.\end{aligned}\tag{42}$$

Letting $l\to\infty$, we arrive at

$$\mathcal{E}_{\rm av}(\mathcal{C})=\mathcal{E}_{\rm av}(\mathcal{X}).\tag{43}$$

□

Theorem 3.3: Suppose that the pair (α,β) of utilization and expected energy per bit is achievable according to Definition 2.6. Then, there exists a family of codes, of strictly increasing lengths and constructed from typical sequences of stationary Markov ergodic processes, that achieves (α,β).

Proof: By assumption, there exists a sequence of codes $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\ldots$ of strictly increasing length which utilize the bus arbitrarily close to a number greater than or equal to α and their expected energy consumptions per bit $\mathcal{E}_b(\mathcal{C}_i)$, $i=1,2,\ldots$ get arbitrary close to a number smaller or equal to β.

We apply Construction II and get a sequence of stationary processes $\mathcal{X}_i=\mathcal{X}(\mathcal{C}_i)$, $i=1,2,\ldots$. By Lemma 3.2, the value $\mathcal{H}(\mathcal{X}_i)/n$ is greater than or equal to the utilization of bus using the code $\mathcal{C}_i$. By Lemma 3.3, the expected energy consumption $\mathcal{E}_{\rm av}(\mathcal{X}_i)$ equals $\mathcal{E}_{\rm av}(\mathcal{C}_i)$. Thus, as $i\to\infty$, $\mathcal{E}_b(\mathcal{X}_i)$ (or a subsequence of it) approaches a number smaller or equal to $\mathcal{E}_b(\mathcal{C}_i)$. By Theorem 3.2, there exist stationary ergodic Markov processes $\mathcal{M}_i$, $i=1,2,\ldots$ whose typical sequences utilize the bus by a factor equal to or larger than $\mathcal{H}(\mathcal{X}_i)/n$ and have expected energy consumption per bit less than or equal to $\mathcal{E}_b(\mathcal{X}_i)$. Combining these observations we conclude the proof. □

Corollary 3.1: For every $\alpha\in(0,1]$, the limiting expected energy consumption per bit $\mathcal{E}_b(\alpha)$ equals $\mathcal{E}_b(\alpha,\boldsymbol{\mathcal{M}_e})$, where $\boldsymbol{\mathcal{M}_e}$ is the set of all stationary ergodic Markov processes in $\mathcal{Q}$.

Proof: The proof follows from Theorems 3.3, Lemma 3.1, and the definition of $\mathcal{E}_b(\alpha)$. □

We can now prove the following important theorem.

Theorem 3.4: For every $\alpha\in(0,1]$, the limiting expected energy consumption per bit $\mathcal{E}_b(\alpha)$ equals $\mathcal{E}_b(\alpha,\mathcal{M})$, where $\mathcal{M}$ is the set of all stationary Markov processes in $\mathcal{Q}$.

Proof: The set of stationary Markov processes, $\mathcal{M}$, has the set of ergodic Markov stationary processes, $\mathcal{M}_{\boldsymbol{e}}$, as a subset and is a subset of the set of stationary processes.

Thus, stationary Markov processes must perform at least as well as stationary ergodic Markov processes and at most as well as stationary processes when considering the tradeoff between utilization and expected energy consumption per bit.

However, we have already proven in Theorem 3.2 that stationary ergodic Markov sources achieve any tradeoff achievable by stationary processes. This means that stationary Markov processes provide the same tradeoff between utilization and energy consumption as that of the stationary processes and stationary ergodic Markov processes. We now apply Corollary 3.1. □

IV. Computation of the Limiting Energy Consumption Per Bit

Theorem 3.4 provides us with the computational arsenal to calculate $\mathcal{E}_b(\alpha)$ by restricting our attention on the class of stationary ergodic Markov sources. To be able to use this arsenal, we will need to study the function $\mathcal{E}_b$ in more details. We start with the following lemma.

Lemma 4.1: For every $\alpha\in(0,1]$, there exists a sequence of codes $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\ldots$ whose bus utilizations tend to α and whose expected energies per bit tend to $\mathcal{E}_b(\alpha)$.

Proof: By the definition of the limiting expected energy per bit, $\mathcal{E}_b(\alpha)$, there exists a sequence of achievable pairs (α_i,β_i) such that $\liminf_{i\to\infty}\alpha_i\ge\alpha$ and $\beta_i\to\mathcal{E}_b(\alpha)$ as $i\to\infty$. Since α_i, $i=1,2,\ldots$ belong to a compact set, there exists a subsequence of achievable pairs $(\hat\alpha_i,\hat\beta_i)$ such that $\hat\alpha_i\to\hat\alpha$ and $\hat\beta_i\to\mathcal{E}_b(\alpha)$ for some $\hat\alpha\ge\alpha$. By an application

of Cantor's diagonalization argument and the definition of achievable pairs, we conclude that there exists a sequence of codes $\hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_3, \ldots$ of strictly increasing lengths, whose bus utilizations tend to $\hat{\alpha}$ and whose expected energies per bit tend to $\mathcal{E}_b(\alpha)$.

Consider the following zero-padding construction. Suppose that a code $\mathcal{C}$ of length L, utilization α, and expected energy per bit $\mathcal{E}_b(\mathcal{C})$ is given. Then, by expanding its codewords by l states of value zero, we obtain a new code $\mathcal{C}^l$ of length $L+l$, utilization $\alpha \frac{L}{L+l}$, and energy per bit $\mathcal{E}_b(\mathcal{C}^l)$ with

$$\frac{L+l}{L+l-1}\left[\mathcal{E}_b(\mathcal{C}) - \frac{\mathcal{E}_{\max}}{\alpha L}\right] \leq \mathcal{E}_b(\mathcal{C}^l) \leq \frac{L+l}{L+l-1}\left[\mathcal{E}_b(\mathcal{C}) + \frac{\mathcal{E}_{\max}}{\alpha L}\right].$$

By applying this zero-padding technique to all the codes in the sequence $\hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_3, \ldots$, we can construct a new sequence of codes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ whose utilization tends to α and whose energy per bit tends to $\mathcal{E}_b(\alpha)$. □

Lemma 4.2: The function $\mathcal{E}_b(\alpha)$ is continuous and nondecreasing for $0 < \alpha \leq 1$.

Proof: The nondecreasing property of $\mathcal{E}_b(\alpha)$ follows directly from its Definition 2.7. We define the function

$$\mathcal{E}_{\text{av}}(\alpha) = n\alpha\mathcal{E}_b(\alpha)$$

and claim that it is continuous. To prove this claim, let $\alpha_1, \alpha_2 \in (0, 1]$ and $\nu \in (0, 1)$ be given. By the definition of achievability and Lemma 4.1, there exist two sequences of codes, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ and $\mathcal{C}_1^*, \mathcal{C}_2^*, \mathcal{C}_3^*, \ldots$, of strictly increasing lengths, whose utilizations tend to α_1 and α_2, respectively, and whose expected energies per bit tend to $\mathcal{E}_b(\alpha_1)$ and $\mathcal{E}_b(\alpha_2)$, respectively. This means that $\mathcal{E}_{\text{av}}(\mathcal{C}_i)$ tends to $n\alpha_1\mathcal{E}_b(\alpha_1) = \mathcal{E}_{\text{av}}(\alpha_1)$ and $\mathcal{E}_{\text{av}}(\mathcal{C}_i^*)$ tends to $n\alpha_2\mathcal{E}_b(\alpha_2) = \mathcal{E}_{\text{av}}(\alpha_2)$.

Now, we use time sharing between $\mathcal{C}_i$ and $\mathcal{C}_i^*$, with shares of ν and $1-\nu$, respectively, to obtain the code $\nu\mathcal{C}_i \circ (1-\nu)\mathcal{C}_i^*$. The number ν is considered rational, $\nu = \frac{\nu_n}{\nu_d}$, and by $\nu\mathcal{C}_i \circ (1-\nu)\mathcal{C}_i^*$ we mean the code formed by the concatenation of ν_n codewords of $\mathcal{C}_i$ and $\nu_d - \nu_n$ codewords of $\mathcal{C}_i^*$. The utilizations of the codes $\nu\mathcal{C}_i \circ (1-\nu)\mathcal{C}_i^*$ tend to $\nu\alpha_1 + (1-\nu)\alpha_2$ and the expected energies per bit, $\mathcal{E}_b(\nu\mathcal{C}_i \circ (1-\nu)\mathcal{C}_i^*)$, of the codes tend to $\frac{\nu\alpha_1\mathcal{E}_b(\alpha_1)+(1-\nu)\alpha_2\mathcal{E}_b(\alpha_2)}{(\nu\alpha_1+(1-\nu)\alpha_2)}$. Thus,

$$\mathcal{E}_b(\nu\alpha_1 + (1-\nu)\alpha_2) \leq \frac{\nu\alpha_1\mathcal{E}_b(\alpha_1) + (1-\nu)\alpha_2\mathcal{E}_b(\alpha_2)}{(\nu\alpha_1 + (1-\nu)\alpha_2)}$$

which establishes the "rational" convexity of $\mathcal{E}_{\text{av}}(\alpha) = n\alpha\mathcal{E}_b(\alpha)$. This, along with the monotonicity of $\mathcal{E}_{\text{av}}(\alpha)$ imply the continuity of $\mathcal{E}_{\text{av}}(\alpha)$ and hence that of $\mathcal{E}_b(\alpha) = \frac{\mathcal{E}_{\text{av}}(\alpha)}{n\alpha}$ in $(0, 1)$. Moreover, since $\mathcal{E}_b(1) < \infty$ and $\mathcal{E}_b$ is an increasing function, Definitions 2.6 and 2.7 imply that

$$\lim_{\alpha \to 1^-} \mathcal{E}_b(\alpha) = \mathcal{E}_b(1).$$ □

For $\alpha \in (0, 1]$ let $\boldsymbol{M}^*(\alpha)$ denote the set of all stationary Markov processes $\mathcal{M}$ whose states are elements of $\mathcal{Q}$ and have entropy rate $\mathcal{H}(\mathcal{M}) \geq \alpha n$.

Lemma 4.3: For every $\alpha \in (0, 1]$ we have

$$\mathcal{E}_b(\alpha) = \inf_{\mathcal{M} \in \boldsymbol{M}^*(\alpha)} \mathcal{E}_b(\mathcal{M}). \tag{44}$$

Proof: Theorem 3.4 implies that

$$\mathcal{E}_b(\alpha) \leq \inf_{\mathcal{M} \in \boldsymbol{M}^*(\alpha)} \mathcal{E}_b(\mathcal{M})$$

as the infimum is taken over a smaller set. Now let $\epsilon > 0$ be given. Applying Theorem 3.4 and the definition of achievability we observe that

$$\inf_{\mathcal{M} \in \boldsymbol{M}^*(\alpha)} \mathcal{E}_b(\mathcal{M}) \leq \mathcal{E}_b(\alpha + \epsilon).$$

This is true since all stationary Markov processes that are candidates for achieving the pair $(\alpha+\epsilon, \mathcal{E}_b(\alpha+\epsilon))$ must have entropy rate higher than αn. Since $\epsilon > 0$ is arbitrary and $\mathcal{E}_b(\cdot)$ is continuous, we conclude that

$$\inf_{\mathcal{M} \in \boldsymbol{M}^*(\alpha)} \mathcal{E}_b(\mathcal{M}) \leq \mathcal{E}_b(\alpha)$$

and the lemma is proven. □

Every pair (P, q) of a $2^n \times 2^n = |\mathcal{Q}| \times |\mathcal{Q}|$ transition matrix P and a 1×2^n state probability row vector q, satisfying $qP = q$, defines a stationary Markov process $\mathcal{M}$. Conversely, every stationary Markov process $\mathcal{M}$ defines, in the obvious way, a pair (P, q) of a transition probability matrix P and a row probability vector q such that $qP = q$. The expected energy consumption per bit and the entropy rate of $\mathcal{M}$ can be computed using the elements of P and q.

From now on, we will mix the notation by identifying the process $\mathcal{M}$ with the pair (P, q). Moreover, we introduce the notation $\mathcal{H}(P, q)$ for $\mathcal{H}(\mathcal{M})$, $\mathcal{E}_{\text{av}}(P, q)$ for $\mathcal{E}_{\text{av}}(\mathcal{M})$, and $\mathcal{E}_b(P, q)$ for $\mathcal{E}_b(\mathcal{M})$. Note that the function

$$\mathcal{E}_b(\mathcal{M}) = \mathcal{E}_b(P, q) = \frac{\mathcal{E}_{\text{av}}(P, q)}{\mathcal{H}(P, q)}$$

is continuous with respect to the elements of the pair (P, q).

Finally, we can interpret the set $\boldsymbol{M}^*(\alpha)$ as the set of all pairs (P, q) with $\mathcal{H}(P, q) \geq \alpha n$. This leads to the following result.

Lemma 4.4: For every $\alpha \in (0, 1]$ we have

$$\mathcal{E}_b(\alpha) = \min_{(P, q) \in \boldsymbol{M}^*(\alpha)} \mathcal{E}_b(P, q). \tag{45}$$

Proof: The function $\mathcal{E}_b(P, q)$ is continuous on the pair (P, q) and the constraints of the problem define a compact set. We thus conclude that then infimum is achieved. □

Now we can state and prove the following theorem that will be important in the calculation of the limiting energy per bit.

Theorem 4.1: For every $\alpha \in (0, 1]$ we have

$$\mathcal{E}_b(\alpha) = \min_{(P, q):\ \mathcal{H}(P, q) = \alpha n} \frac{\mathcal{E}_{\text{av}}(P, q)}{\alpha n}. \tag{46}$$

Proof: To prove the theorem, we will construct a minimizing pair (P, q) for (45) in Lemma 4.4 such that $\mathcal{H}(P, q) = \alpha n$. Once this is established, the result follows.

Suppose that (P_*, q_*) is a minimizing pair for (45) with $\mathcal{H}(P_*, q_*) > \alpha n$. For $\theta \in [0, 1]$ we define the pair

$$\begin{aligned}(P_\theta, q_*) &= \theta(P_*, q_*) + (1-\theta)(I, q_*) \\ &= (\theta P_* + (1-\theta)I, q_*)\end{aligned}$$

where I is the $2^n \times 2^n$ identity matrix. The new pair satisfies the equality $q_* P_\theta = q_*$. Also, we have[2]

$$\begin{aligned}\mathcal{E}_{\rm av}(P_\theta, q_*) &= \mathcal{E}_{\rm av}(\theta P_* + (1-\theta)I, q_*) \\ &= \theta \mathcal{E}_{\rm av}(P_*, q_*) + (1-\theta)\mathcal{E}_{\rm av}(I, q_*) \\ &= \theta \mathcal{E}_{\rm av}(P_*, q_*) + 0.\end{aligned}$$

The concavity of the function $\mathcal{H}(\cdot, q_*)$ implies that

$$\begin{aligned}\mathcal{H}(P_\theta, q_*) &= \mathcal{H}(\theta P_* + (1-\theta)I, q_*) \\ &\geq \theta \mathcal{H}(P_*, q_*) + (1-\theta)\mathcal{H}(I, q_*) \\ &= \theta \mathcal{H}(P_*, q_*) + 0.\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{E}_b(P_\theta, q_*) &= \frac{\mathcal{E}_{\rm av}(P_\theta, q_*)}{\mathcal{H}(P_\theta, q_*)} \\ &\leq \frac{\theta \mathcal{E}_{\rm av}(P_*, q_*)}{\theta \mathcal{H}(P_*, q_*)} = \mathcal{E}_b(P_*, q_*).\end{aligned}$$

Since $\mathcal{H}(P_0, q_*) = \mathcal{H}(I, q_*) = 0$, there exists some θ' such that $\mathcal{H}(P_{\theta'}, q_*) = \alpha n$. We conclude that $(P_{\theta'}, q_*)$ is also a minimizing pair of (45) with $\mathcal{H}(P_{\theta'}, q_*) = \alpha n$. □

In light of Theorem 4.1, the computation of $\mathcal{E}_b(\alpha)$ has been reduced to a constrained minimization problem that can be solved using standard methods such as the application of Lagrange multipliers.

In the following, we will prove that for every $\alpha \in (0, 1)$ there is a stationary Markov process that achieves the minimum in the aforementioned problems. We will derive formulas for the limiting expected energy as a function of the utilization.

Let (P, q) be a pair of a stochastic matrix and a probability eigenvector of it. The pair defines the matrix $\Pi = [\pi_{i,j}]_{i,j=0}^{2^n-1}$ with $\pi_{i,j} = q_i P_{i,j}$. For every i, j, the entries $\pi_{i,j}$ satisfy the following relations:

$$\pi_{i,j} \geq 0 \tag{47}$$

$$\sum_{i,j} \pi_{i,j} = 1 \tag{48}$$

and

$$\sum_j \pi_{i,j} = \sum_k \pi_{k,i}. \tag{49}$$

The last equality holds because

$$\sum_j \pi_{i,j} = \sum_j q_i P_{i,j} = q_i = \sum_k q_k P_{k,i} = \sum_k \pi_{k,i}$$

and can be written also as

$$\sum_j (\pi_{i,j} - \pi_{j,i}) = 0.$$

The entropy of the matrix Π is defined as

$$\mathcal{H}(\Pi) = -\sum_{i,j} \pi_{i,j} \log\left(\frac{\pi_{i,j}}{\sum_k \pi_{i,k}}\right) \tag{50}$$

and, of course, equals $\mathcal{H}(P, q)$. (We agree that $0 \log \frac{0}{0} = 0$ and a zero row of matrix Π contributes zero to the entropy.) Now, independently of the pair (P, q), we define $\boldsymbol{\Pi}$ to be the set of all $2^n \times 2^n$ matrices Π satisfying the constraints (47)–(49). We also define $\boldsymbol{\Pi}_+$ as the subset of $\boldsymbol{\Pi}$ consisting of only the positive matrices. We have the following lemma.

[2]Here we assume that $\mathcal{E}(\sigma \to \sigma) = 0$ for all $\sigma \in \mathcal{Q}$. This is true for the energy cost function (1).

Lemma 4.5: The mapping

$$\Pi \longmapsto P = \left[\frac{\pi_{i,j}}{\sum_k \pi_{i,k}}\right]_{i,j}$$

from $\boldsymbol{\Pi}_+$ to the subset of positive stochastic matrices is bijective.

Proof: Suppose that the positive stochastic matrix P is the image of an element $\Pi' = [\pi'_{i,j}]_{i,j}$ of $\boldsymbol{\Pi}_+$. Then, there exists a unique positive vector $q' = [q'_i]_i$ such that $\pi'_{i,j} = q'_i P_{i,j}$. It is straightforward to verify that $q'_i = \sum_k \pi'_{i,k}$. To show that the mapping $\Pi \longmapsto P$ is injective it suffices to show that there is only one positive vector q for which $[q_i P_{i,j}]_{i,j}$ belongs to $\boldsymbol{\Pi}_+$. For vector q, (49) implies

$$\sum_i q_i P_{i,j} = \sum_k q_j P_{j,k} = q_j. \tag{51}$$

Therefore, q must be a positive left eigenvector of P. The matrix P is positive and, by Perron's theorem, q is unique up to a positive factor. The vector q is uniquely defined because of (48) which implies $\sum_i q_i = \sum_{i,j} q_i P_{i,j} = 1$.

To show the mapping is onto, we start with a positive stochastic matrix P and set $\pi_{i,j} = v_i P_{i,j}$, where v is the unique (left) probability eigenvector of P. Then $\sum_{i,j} \pi_{i,j} = 1$ and

$$\sum_k \pi_{k,i} = \sum_k v_k P_{k,i} = v_i = \sum_j v_i P_{i,j} = \sum_j \pi_{i,j}.$$

Therefore, Π belongs to $\overline{\boldsymbol{\Pi}}_+$. □

In the following, we will write $\Pi = (P, q)$ to denote that $\pi_{i,j} = q_i P_{i,j}$. Note that for a matrix $\Pi \in \overline{\boldsymbol{\Pi}}$ with a zero row, there is a set of pairs (P, q) such that $\Pi = (P, q)$. All of these pairs have the same vector q, the entries of which correspond to zero rows of Π, must be zero.

Lemma 4.6: The set $\overline{\boldsymbol{\Pi}}$ is convex. The entropy function $\mathcal{H}$ is concave in $\overline{\boldsymbol{\Pi}}$ and strictly concave in the subset $\overline{\boldsymbol{\Pi}}_+$.

Proof: Conditions (47)–(49), directly imply the convexity of the set $\overline{\boldsymbol{\Pi}}$. Now let Π^1 and Π^2 be in $\overline{\boldsymbol{\Pi}}$ and a be a constant such that $0 < a < 1$. Set $\Pi = a\Pi^1 + (1-a)\Pi^2$. The *log-sum* inequality states that

$$\begin{aligned}&a\pi^1_{i,j} \log\left(\frac{a\pi^1_{i,j}}{a\sum_k \pi^1_{i,k}}\right) + (1-a)\pi^2_{i,j} \log\left(\frac{(1-a)\pi^2_{i,j}}{(1-a)\sum_k \pi^2_{i,k}}\right) \\ &\geq \left(a\pi^1_{i,j} + (1-a)\pi^2_{i,j}\right) \log\left(\frac{a\pi^1_{i,j} + (1-a)\pi^2_{i,j}}{a\sum_k \pi^1_{i,k} + (1-a)\sum_k \pi^2_{i,k}}\right)\end{aligned}$$

with equality if and only if

$$\frac{a\pi^1_{i,j}}{a\sum_k \pi^1_{i,k}} = \frac{(1-a)\pi^2_{i,j}}{(1-a)\sum_k \pi^2_{i,k}}.$$

Therefore, we have

$$\begin{aligned}\mathcal{H}(\Pi) = &-\sum_{i,j} \left(a\pi^1_{i,j} + (1-a)\pi^2_{i,j}\right) \\ &\cdot \log\left(\frac{a\pi^1_{i,j} + (1-a)\pi^2_{i,j}}{a\sum_k \pi^1_{i,k} + (1-a)\sum_k \pi^2_{i,k}}\right)\end{aligned}$$

$$\geq -a\sum_{i,j}\pi^1_{i,j}\log\left(\frac{\pi^1_{i,j}}{\sum_k \pi^1_{i,k}}\right) - (1-a)\sum_{i,j}\pi^2_{i,j}\log\left(\frac{\pi^2_{i,j}}{\sum_k \pi^2_{i,k}}\right) = a\mathcal{H}(\Pi^1)+(1-a)\mathcal{H}(\Pi^2)$$

and so $\mathcal{H}(\Pi)\geq a\mathcal{H}(\Pi^1)+(1-a)\mathcal{H}(\Pi^2)$. Suppose now that Π^1 and Π^2 have positive elements, and, therefore, belong to $\mathbf{\Pi}_+$. In this case, the equality above holds if and only if $P^1=P^2$ where

$$P^r=\left[\frac{\pi^r_{i,j}}{\sum_k \pi^r_{i,k}}\right]_{i,j}$$

and $r=1,2$. Applying Lemma 4.5, we conclude that $\mathcal{H}(\Pi)=a\mathcal{H}(\Pi^1)+(1-a)\mathcal{H}(\Pi^2)$ if and only if $\Pi^1=\Pi^2$. □

The expected energy and expected energy per bit corresponding to the elements of the set $\mathbf{\Pi}$ are defined analogously to those of previous sections. This is done by letting

$$\mathcal{E}_{\rm av}(\Pi)=\sum_{i,j}\pi_{i,j}\mathcal{E}(i\to j)$$

and

$$\mathcal{E}_b(\Pi)=\mathcal{E}_{\rm av}(\Pi)/\mathcal{H}(\Pi).$$

Note also that if $(P,q)=\Pi$ then $\mathcal{E}_{\rm av}(P,q)=\mathcal{E}_{\rm av}(\Pi)$ and $\mathcal{E}_b(P,q)=\mathcal{E}_b(\Pi)$. It is straightforward to verify that

$$\mathcal{E}_b(\alpha)=\min_{\Pi\in\mathbf{\Pi},\,\mathcal{H}(\Pi)\geq an}\mathcal{E}_b(\Pi). \tag{52}$$

If (P,q) is an optimal point for

$$\mathcal{E}_b(\alpha)=\min_{(P,q)\in\boldsymbol{M}^*(\alpha)}\mathcal{E}_b(P,q)$$

then the minimizing Π in (52) is achieved by $\Pi=[q_iP_{i,j}]_{i,j}$. Conversely, if $\Pi\in\mathbf{\Pi}$ achieves the minimum in (52) and $(P,q)=\Pi$ then (P,q) is the minimizing of

$$\mathcal{E}_b(\alpha)=\min_{(P,q)\in\boldsymbol{M}^*(\alpha)}\mathcal{E}_b(P,q).$$

Lemma 4.7: Every solution Π of

$$\mathcal{E}_b(\alpha)=\min_{\Pi\in\mathbf{\Pi},\,\mathcal{H}(\Pi)\geq an}\mathcal{E}_b(\Pi)$$

belongs to $\mathbf{\Pi}_+$.

Proof: Let Π be a solution of (52). We show first that if for some state k we have $\sum_j\pi_{k,j}>0$, then $\pi_{k,j}>0$ for all j. Suppose that this is not the case and let r be such that $\pi_{k,r}=0$. For $\delta\in(0,\frac{1}{2})$, we set

$$\Pi_\delta=(1-2\delta)\Pi+\delta e_ke_r^T+\delta e_re_k^T$$

where e_i is the vector with one in the ith coordinate and zeros everywhere else. It is easy to verify that $\Pi_\delta\in\mathbf{\Pi}$ for every δ. If $\pi^\delta_{i,j}$ denotes the i,j element of Π_δ, then

$$\mathcal{H}(\Pi_\delta)=-\sum_{i,j}\pi^\delta_{i,j}\log\left(\frac{\pi^\delta_{i,j}}{\sum_v\pi^\delta_{i,v}}\right).$$

For every i,j we have

$$\pi^\delta_{i,j}\log\left(\frac{\pi^\delta_{i,j}}{\sum_v\pi^\delta_{i,v}}\right)=[(1-2\delta)\pi_{i,j}+\delta(\delta_{i,k}\delta_{j,r}+\delta_{i,r}\delta_{j,k})]\cdot\log\left(\frac{(1-2\delta)\pi_{i,j}+\delta(\delta_{i,k}\delta_{j,r}+\delta_{i,r}\delta_{j,k})}{(1-2\delta)\sum_v\pi_{i,v}+\delta(\delta_{i,k}+\delta_{i,r})}\right)$$

where $\delta_{i,j}$ is 1 if $i=j$ and 0 otherwise. From the above expression we can conclude that for every i,j we have

$$\pi^\delta_{i,j}\log\left(\frac{\pi^\delta_{i,j}}{\sum_v\pi^\delta_{i,v}}\right)=\pi_{i,j}\log\left(\frac{\pi_{i,j}}{\sum_v\pi_{i,v}}\right)+c_{i,j}\delta\log\delta+\mathcal{O}(\delta)$$

with $c_{i,j}$ being a nonnegative constant ($c_{i,j}=0$, 1 or 2). Now, according to our assumptions it is $\pi_{k,r}=0$ while $\sum_j\pi_{k,j}>0$. This implies that

$$\pi^\delta_{k,r}\log\left(\frac{\pi^\delta_{k,r}}{\sum_v\pi^\delta_{k,v}}\right)=0+c_{k,r}\delta\log\delta+\mathcal{O}(\delta)$$

with $c_{k,r}=2$ or $c_{k,r}=1$ depending on whether $k=r$ or not. We conclude that there exists a positive constant $c_1\geq 1$ such that for sufficiently small δ

$$\mathcal{H}(\Pi_\delta)=\mathcal{H}(\Pi)-c_1\delta\log\delta+\mathcal{O}(\delta).$$

Also, because $\mathcal{E}_{\rm av}(\Pi)$ depends linearly on Π, there is a constant c_2 such that

$$\mathcal{E}_{\rm av}(\Pi_\delta)=\mathcal{E}_{\rm av}(\Pi)+c_2\delta.$$

Note that $\mathcal{H}(\Pi)$ is positive since $\mathcal{H}(\Pi)\geq an$. Also, $\mathcal{E}_{\rm av}(\Pi)$ is positive since the transition energy $\mathcal{E}(\sigma\to\omega)$ is zero only[3] when $\sigma=\omega$. Therefore, setting $c_3=c_2/\mathcal{E}_{\rm ave}(\Pi)$ and $c_4=c_1/\mathcal{H}(\Pi)>0$ we have

$$\frac{\mathcal{E}_{\rm av}(\Pi_\delta)}{\mathcal{H}(\Pi_\delta)}=\frac{\mathcal{E}_{\rm av}(\Pi)}{\mathcal{H}(\Pi)}\cdot\frac{1+c_3\delta}{1-c_4\delta\log\delta+\mathcal{O}(\delta)}$$

with $c_4>0$. This means that there exists an arbitrary small δ for which $\mathcal{E}_b(\Pi_\delta)<\mathcal{E}_b(\Pi)$ and $\mathcal{H}(\Pi_\delta)\geq\mathcal{H}(\Pi)=an$. A contradiction! Note that a state k with $\sum_j\pi_{k,j}>0$ always exists. Hence, there exists a row of Π with only positive entries. This means that all column-sums of Π are positive and because of property (49), all row-sums are positive as well. Using the argument above we conclude that Π is a positive matrix. □

The above lemmas lead to the following theorem.

Theorem 4.2: Problem (52) has a unique solution Π. The solution is positive, $\Pi\in\mathbf{\Pi}_+$, and with $\mathcal{H}(\Pi)=an$. There is no $\alpha\in(0,1)$ for which the solution equals the matrix $\Omega=\frac{1}{2^{2n}}[1]_{i,j}$.

Proof: From Lemma 4.7, we know that every solution of (52) is positive. Linearity of the expected energy function $\mathcal{E}_{\rm av}(\Pi)$ and strict concavity of $\mathcal{H}(\Pi)$ in $\mathbf{\Pi}_+$ imply the uniqueness of the solution. The same properties also imply that $\mathcal{H}(\Pi)=an$. This can be shown in a way similar to that

[3]Here we need to assume that $\mathcal{E}(\sigma\to\omega)=0$ only when $\sigma=\omega$. This is satisfied by the energy cost function (1).

of Theorem 4.1. Finally, $\mathcal{H}(\Omega) = n$ results in $\Pi \neq \Omega$ since $\alpha n < n$. □

The theorem has the following immediate consequences.

Corollary 4.1: The minimizing pair (P, q) in (45) exists and is unique. Furthermore, the transition probability matrix P and the probability eigenvector q are positive.

Corollary 4.2: Problem (53) has a unique solution. Its solution is positive and identical to that of (52).

$$\mathcal{E}_b(\alpha) = \min_{\Pi \in \mathbf{\Pi},\, \mathcal{H}(\Pi) = an} \frac{1}{n\alpha}\, \mathcal{E}_{\text{av}}(\Pi). \tag{53}$$

Lemma 4.8: The (unique) solution of (52) and (53) is a regular point of the set of (active) constraints of (53), which are: $\sum_{i,j} \pi_{i,j} = 1$, $\sum_j \pi_{i,j} = \sum_k \pi_{k,i}$ for every i, and $\mathcal{H}(\Pi) = an$.

Proof: The gradients of the constraints are given in matrix form as follows:

$$B = \left[\frac{\partial \left(\sum_{i,j} \pi_{i,j} \right)}{\partial \pi_{k,r}} \right]_{k,r} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$A_i = \left[\frac{\partial \left(\sum_j (\pi_{i,j} - \pi_{j,i}) \right)}{\partial \pi_{k,r}} \right]_{k,r}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix}$$

where the nonzero elements of matrices A_i are in the ith row and the ith column. And finally

$$C = -\left[\frac{\partial \mathcal{H}(\Pi)}{\partial \pi_{k,r}} \right]_{k,r} = \left[\log \left(\frac{\pi_{k,r}}{\sum_v \pi_{k,v}} \right) \right]_{k,r}.$$

The set $\{A_i\}_i$ is linearly independent and B is orthogonal to every A_i under the inner product $\langle X, Y \rangle = \text{tr}(XY')$. We conclude that any nontrivial linear dependence between the matrices A_i, B, and C can be written as $C = bB + \sum_k r_k A_k$. Element-wise this is equivalent to $c_{i,j} = b + r_i - r_j$ or $P_{i,j} = \exp(b + r_i - r_j)$, with

$$P = \left[\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} \right]_{i,j}.$$

Since for every state i, $\sum_j P_{i,j} = 1$ we conclude that $r_i = r_j$ for all i, j and, therefore, $P = \frac{1}{2^n}[1]_{i,j}$. This implies $\Pi = [\frac{1}{2^{2n}}]_{i,j} = \Omega$ as the only irregular point of the constraints. The proof follows from an application of Theorem 4.2. □

The fact that the optimal Π satisfies $\mathcal{H}(\Pi) = an$ and $\Pi \in \mathbf{\Pi}_+$ combined with the smoothness of the constraints within $\mathbf{\Pi}_+$ gives the following result.

Corollary 4.3: The solution of problem (53) is a stationary point of its *Lagrangian*

$$\mathcal{L} = v \sum_{i,j} \pi_{i,j} + \sum_{i,j} \lambda_i (\pi_{i,j} - \pi_{j,i}) - \mu \sum_{i,j} \pi_{i,j} \ln \left(\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} \right) + \sum_{i,j} \pi_{i,j} \mathcal{E}(i \to j). \tag{54}$$

Proof: From Lemma 4.8, the solution Π of the problem is a regular point of its constraints. Therefore, there exists a set of real numbers v, λ_i, μ for which $\frac{\partial \mathcal{L}}{\partial \pi_{i,j}} = 0$ for all i, j (see [3, Proposition 3.1.1, p. 255]). □

Lemma 4.9: Every stationary point Π of the *Lagrangian* $\mathcal{L}$ in the subset $\mathbf{\Pi}_+$ is of the form

$$\Pi = \left[\frac{g_i g_j e^{-\gamma \mathcal{E}(i \to j)}}{\sum_{r,k} g_r g_k e^{-\gamma \mathcal{E}(r \to k)}} \right]_{i,j} \tag{55}$$

where $g = (g_0, g_1, \ldots, g_{2^n - 1})'$ is the unique (up to a constant) positive eigenvector of the matrix $W(\gamma) = [e^{-\gamma \mathcal{E}(i \to j)}]_{i,j}$ and γ is a real number.

Proof: The partial derivative of the Lagrangian with respect to the variable $\pi_{i,j}$ is

$$\frac{\partial \mathcal{L}}{\partial \pi_{i,j}} = v + \lambda_i - \lambda_j - \mu \ln \left(\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} \right) + \mathcal{E}(i \to j). \tag{56}$$

We are interested in the solutions of the set of equations $\frac{\partial \mathcal{L}}{\partial \pi_{i,j}} = 0$, i.e.,

$$v + \lambda_i - \lambda_j - \mu \ln \left(\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} \right) + \mathcal{E}(i \to j) = 0. \tag{57}$$

First we examine the case $\mu = 0$ and show that it is not feasible. Suppose that $\mu = 0$. Then (57) implies that

$$\mathcal{E}(i \to j) = -v - \lambda_i + \lambda_j, \qquad \text{for every } i, j.$$

Since the energy cost function, (1), is symmetric, that is, $\mathcal{E}(i \to j) = \mathcal{E}(j \to i)$, we must have $\lambda_i = \lambda_j$ for every i, j. Therefore, $\mathcal{E}(i \to j) = -v$, i.e., the cost function is constant. But this is impossible since, from (1), we have $\mathcal{E}(i \to i) = 0$ and $\mathcal{E}(i \to j) \neq 0$ for every $i \neq j$ (matrix $\mathcal{A}$ given by (2) is positive definite). Therefore, $\mu \neq 0$, and the system of equations (57) implies that

$$\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} = e^{\frac{v + \lambda_i - \lambda_j + \mathcal{E}(i \to j)}{\mu}}. \tag{58}$$

Since all the parameters v, λ_i, and μ are real we can do the transformation: $f = e^{v/\mu}$, $g_i = e^{-\lambda_i/\mu}$, and $\gamma = -1/\mu$, and have $f > 0$ and $g_i > 0$ for every i. Then, (58) becomes

$$\frac{\pi_{i,j}}{\sum_k \pi_{i,k}} = f\, \frac{g_j}{g_i}\, e^{-\gamma \mathcal{E}(i \to j)}. \tag{59}$$

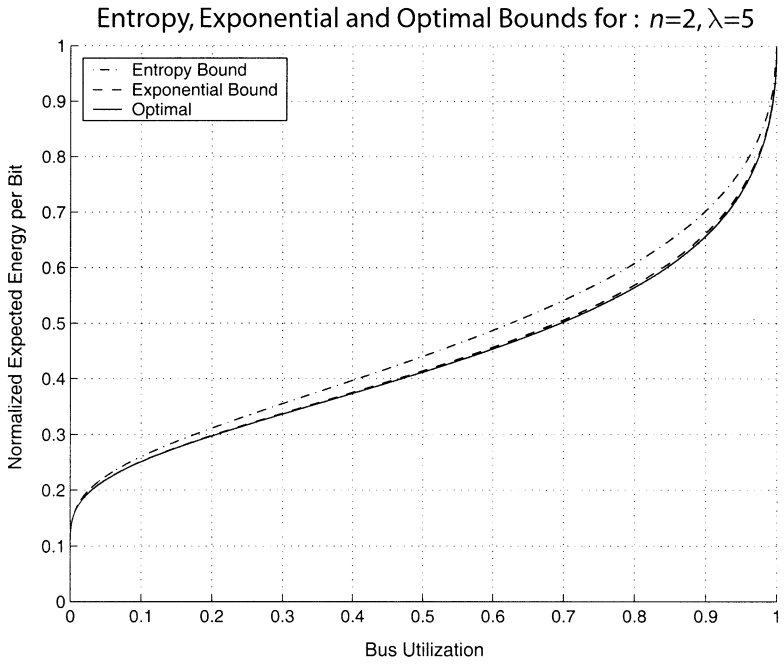

Fig. 5. Entropy, exponential, and optimal bounds for $n = 2$ and $\lambda = 5$.

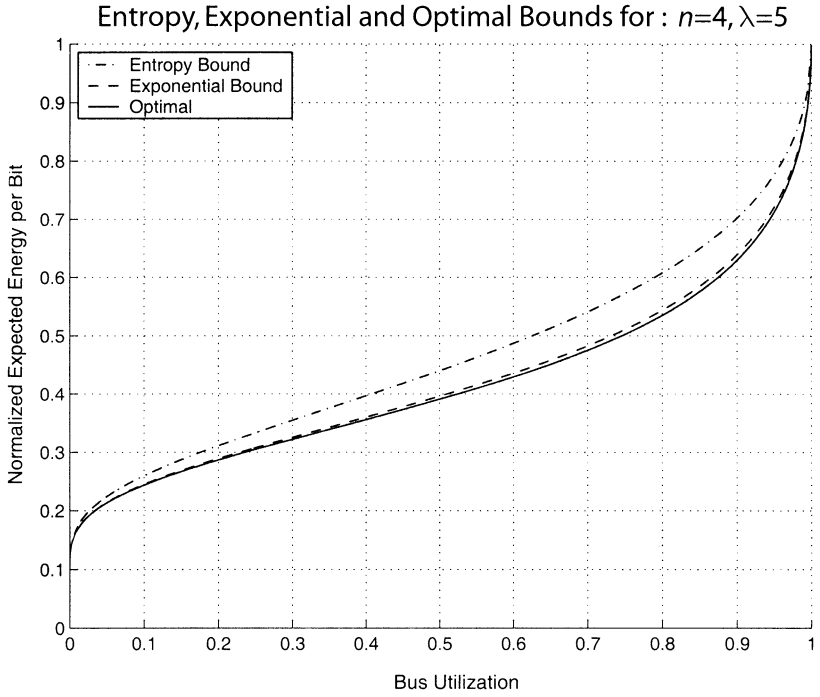

Fig. 6. Entropy, exponential, and optimal bounds for $n = 4$ and $\lambda = 5$.

Summing over j, we get $g_i = f \sum_j e^{-\gamma \mathcal{E}(i \to j)} g_j$ or written in matrix form, $g = fWg$ with $g = (g_0, g_1, \ldots, g_{2^n-1})'$ and $W = [e^{-\gamma \mathcal{E}(i \to j)}]_{i,j}$. The matrix fW is always positive and, therefore, g must be its unique (up to a factor) positive eigenvector [10, Theorem 4.4, p. 16]. Therefore, f must be the inverse of the maximal eigenvalue $\lambda_{\max}$ of W. Recall that the energy cost function is symmetric, i.e., $\mathcal{E}(i \to j) = \mathcal{E}(j \to i)$ for every i, j. Therefore, we also have that $g' = fg'W$ or more explicitly

$$g_j = f \sum_i g_i e^{-\gamma \mathcal{E}(i \to j)}. \tag{60}$$

We define the probability vector

$$q = (g_0^2, g_1^2, \ldots, g_{2^n-1}^2)/||g||^2$$

and using (60) we show that it is a left eigenvector of P

$$\sum_i q_i P_{i,j} = \sum_i \frac{g_i^2}{||g||^2} f \frac{g_j}{g_i} e^{-\gamma \mathcal{E}(i \to j)}$$

$$= \frac{g_j}{||g||^2} f \sum_i g_i e^{-\gamma \mathcal{E}(i \to j)} = \frac{g_j^2}{||g||^2} = q_j.$$

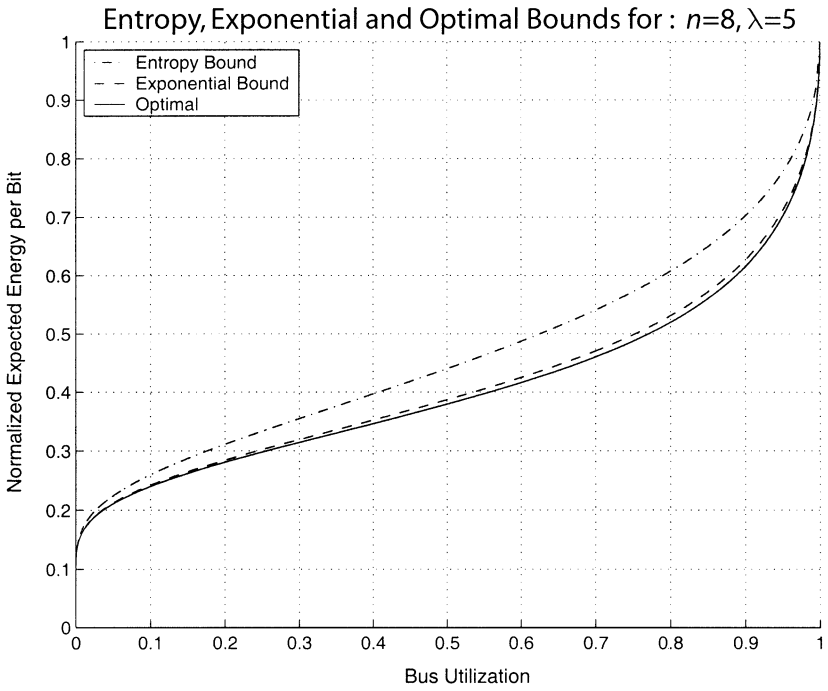

Fig. 7. Entropy, exponential, and optimal bounds for $n = 8$ and $\lambda = 5$.

Applying Lemma 4.5, we get $\Pi = (P, q)$. This means that every stationary point Π of the Lagrangian must have entries $\pi_{i,j}$ of the form

$$\pi_{i,j} = \frac{g_i g_j}{||g||^2} f e^{-\gamma \mathcal{E}(i \to j)}.$$

This concludes the proof of the lemma. □

Note that for every γ the matrix W is positive and so it has a unique positive maximal eigenvalue. The matrix W is also analytic in γ. Therefore, its eigenvalue and the corresponding normalized eigenvector are both analytic functions of γ [7, Theorem 1, p. 396]. Even more, the stationary point Π of the Lagrangian, parameterized on γ, approaches the identity matrix as γ approaches $+\infty$. This is because only transitions from every state to itself have zero cost. Also, Π approaches Ω as γ approaches zero. We conclude as follows.

Theorem 4.3: For every $\alpha \in (0, 1)$, there exists a unique positive γ for which matrix Π, defined by (55), satisfies $\mathcal{H}(\Pi) = \alpha n$ and is the unique solution of problems (52) and (53). The entropy $\mathcal{H}(\Pi)$ is a strictly decreasing function of γ for $\gamma > 0$.

Theorem 4.3 can be proved using well-known results on superconvex functions and properties of eigenvalues of Hadamard matrix exponentials. The proof is lengthy, technical, and is omitted.

Theorem 4.3 is the tool to evaluate the limiting expected energy per transmitted bit, $\mathcal{E}_b(\alpha)$. Figs. 5–7 present the normalized value $\mathcal{E}_b(\alpha)/\mathcal{E}_u$, named the *optimal bound*, as a function of the bus utilization α, for $\lambda = 5$ and $n = 2, 4, 8$. For comparison, we have included the exponential and the entropy bounds.

V. Conclusion and Final Remarks

We have considered the problem of reducing the energy consumption (or other computation cost functions) in a computation module as a coding and information theory problem. It has been shown that redundancy in the capacity of the computation module can be exploited, using coding techniques, to achieve significant reductions in the energy consumption (or other computation cost). Redundancy can be introduced by adding extra ports in the module, or by reducing the actual information rate in the input, and by coding the input and output sequences. Using tools of information theory, we have derived the maximum possible cost *reduction* in an analytic form that can be used for numerical calculations.

Although, for the ease of presentation, we used a specific cost function, it is noted that the methods we established here apply to a very general class of cost functions for which the cost of a sequence of computations is the sum of the costs of the individual steps. Some related issues and a more explicit form of $\mathcal{E}_b(\cdot)$ can be founs in [25]–[27].

References

[1] V. Agarwal, M. S. Hrishikesh, S. W. Keckler, and D. Burger, "Clock rate versus IPC: The end of the road for conventional microarchitectures," in *Proc. 27th Int. Symp. Computer Architecture*, 2000, pp. 248–259.

[2] Benini, G. De Micheli, E. Macii, D. Sciuto, and C. Silvano, "Asymptotic zero-transition activity encoding for address busses in low-power microprocessor-based systems," in *Proc. Great Lakes VLSI Symp.*, 1997, pp. 77–82.

[3] D. P. Bertsekas, *Nonlinear Programming*, 2nd ed. Belmont, MA: Athena Scientific, 1995.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[5] R. J. Fletcher, "Integrated circuit having outputs configured for reduced state changes," U.S. Patent 4.667.337, May 1987.

[6] K. W. Kim, K. H. Baek, N. Shanbhag, C. L. Liu, and S. M. Kang, "Coupling-driven signal encoding scheme for low-power interface design," in *Proc. Int. Conf. Computer Aided Design*, San Jose, CA, Nov. 2000, pp. 318–321.

[7] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed. New York: Academic, 1988.

[8] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1996.

[9] D. Marculescu, R. Marculescu, and M. Pedram, "Information theoretic measures for power analysis," *IEEE Trans. Computer-Aided Design*, vol. 15, pp. 599–610, June 1996.

[10] H. Minc, *Nonnegative Matrices (Wiley Interscience Series in Discrete Mathematics and Optimization)*. New York: Wiley, 1988.

[11] F. N. Najm, "Transition density: A new measure of activity in digital circuits," *IEEE Trans. Computer-Aided Design* , vol. 12, pp. 310–323, Feb. 1993.

[12] S. Ramprasad, N. R. Shanbhag, and I. N. Hajj, "Information-theoretic bounds on average signal transition activity," *IEEE Trans. VLSI Syst.*, vol. 7, pp. 359–368, Sept. 1999.

[13] ——, "A coding framework for low-power address and data busses," *IEEE Trans. VLSI Syst.*, vol. 7, pp. 212–221, June 1999.

[14] S. A. Savari and R. G. Gallager, "Arithmetic coding for finite-state noiseless channels," *IEEE Trans. Inform. Theory*, vol. 40, pp. 100–107, Jan. 1994.

[15] S. A. Savari, "A probabilistic approach to some asymptotics in noiseless communication," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1246–1262, July 2000.

[16] N. R. Shanbhag, "A mathematical basis for power reduction in digital VLSI systems," *IEEE Trans. Circuits Syst. II*, vol. 44, pp. 935–951, Nov. 1997.

[17] P. Sotiriadis and A. Chandrakasan, "A bus energy model for deep submicron technology," *IEEE Trans. VLSI Syst.*, vol. 10, pp. 341–350, June 2002.

[18] ——, "Low power bus coding techniques considering inter-wire capacitances," in *Proc. Custom Integrated Circuits Conf. (CICC 2000)*, Orlando, FL, May 2000, pp. 507–510.

[19] P. Sotiriadis, A. Wang, and A. Chandrakasan, "Transition pattern coding: An approach to reduce energy in interconnect," in *Proc. 26 European Solid-State Circuits Conf.* , Stockholm, Sweden, Sept. 2000, pp. 320–323.

[20] P. Sotiriadis and A. Chandrakasan, "Bus energy minimization by transition pattern coding (TPC) in deep sub-micron technologies," in *Proc. Int. Conf. Computer Aided Design*, San Jose, CA, Nov. 2000, pp. 322–327.

[21] S. K. Srinivasan and K. M. Mehta, *Stochastic Processes*. New Delhi, India: Tata McGraw-Hill, 1988.

[22] M. R. Stan and W. P. Burleson, "Limited weight codes for low-power I/0," in *Proc. IWLPD*, 1994, pp. 209–215.

[23] M. R. Stan and W. P. Burleson, "Bus-invert coding for low-power I/0," *IEEE Trans. VLSI Syst.*, vol. 3, pp. 49–58, Mar. 1995.

[24] ——, "Low-power encodings for global communication in CMOS VLSI," *IEEE Trans. VLSI Syst.*, vol. 5, pp. 444–455, Dec. 1997.

[25] P. Sotiriadis, "Interconnect modeling and optimization in deep sub-micron technologies," Ph.D. dissertation, MIT, Cambridge, MA, May 2002.

[26] P. Sotiriadis and A. Chandrakasan, "Power estimation and power and statistical measures," *J. Circuits, Syst., Comput.*, Mar. 2003, to be published.

[27] P. Sotiriadis, "Energy reduction coding," presented at the 37th Conference on Information Sciences and Systems, Baltimore, MD, Mar. 2003.