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Single-$U\overline{RC}$ Integrators

Paul Peter Sotiriadis and Yannis Tsividis

***Abstract*—This brief proves that single-pole behavior can be achieved using a single grounded uniformly distributed RC element. The transfer function generated out of a grounded $U\overline{RC}$ and a memoryless linear feedback network is studied leading to two novel integrator topologies.**

***Index Terms*—Active filter, distributed, feedback, integrator, irrational transfer function, transmission line, topology.**

I. Introduction

Linear distributed RC elements have been studied for several decades. Many different approaches have been proposed to exploit their dynamic behavior in order to compose meaningful types of impedances, transfer functions and, of course, filters. Filter synthesis techniques using distributed RCs can be classified into four general groups.

1) Exact synthesis of transfer functions rational on the $\Lambda = \tanh(\alpha\sqrt{s}\,)$ plane [1] or the $P = \cosh(\alpha\sqrt{s}\,)$ plane [2].
2) Exact transfer function synthesis, rational on the s plane, using nonuniform or/and nongrounded or/and multi-layer distributed structures, e.g., [3], [4].
3) Exact transfer function synthesis, rational on the s plane, using pairs of uniform commensurate (i.e., of the same time constant) distributed RCs, e.g., [5], [6].
4) Techniques leading to transfer functions approximately rational on the s plane. (An excellent source of results and references is [7].

This brief proves that we can build an integrator using a single grounded uniformly distributed RC element ($U\overline{RC}$) along with a linear, memoryless network. This avoids the need for exactly commensurate pairs [5], [6] or more complex structures [3], [4]. The dynamic behavior of a grounded $U\overline{RC}$ in a memoryless lineal feedback network is studied leading to two novel integrator topologies.

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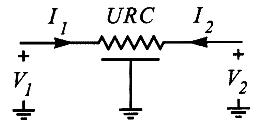

Fig. 1. Grounded $\boldsymbol{U\overline{RC}}$.

II. Grounded Uniformly Distributed RC Line ($U\overline{RC}$)

A grounded $U\overline{RC}$ is a symmetric two-port linear element characterized by its resistance per-unit length R_0 in Ω/m, its capacitance per-unit length C_0 in F/m and its total length L. It is symbolically represented by the T network of Fig. 1.

The total resistance and capacitance are $R = R_0 L$ and $C = C_0 L$, respectively. The time constant τ is defined as

$$\tau = R_0 C_0 L^2 = RC \tag{1}$$

and is a measure of the propagation delay along the body of the $U\overline{RC}$. For frequencies much smaller than $1/\tau$ the $U\overline{RC}$ behaves like a lumped RC element. The $U\overline{RC}$ accepts all two-port descriptions [7]; in particular, if Z_0 is its driving impedance and Z_m is its transimpedance, we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_0 & Z_m \\ Z_m & Z_0 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{2}$$

with [7]

$$\begin{aligned} Z_0(s) &= \frac{\sqrt{\tau s}\coth\left(\sqrt{\tau s}\,\right)}{Cs} \\ Z_m(s) &= \frac{\sqrt{\tau s}\,\mathrm{csch}\left(\sqrt{\tau s}\,\right)}{Cs}. \end{aligned} \tag{3}$$

Although Z_0 and Z_m are both irrational functions of s, it can be verified directly that they satisfy

$$Z_0^2 - Z_m^2 = \frac{R}{Cs}. \tag{4}$$

This relation involves the squares of the impedances, so to realize it we need to use two $U\overline{RC}$s, or one $U\overline{RC}$ in a feedback loop (as it will be shown in Section III). Note also that the left part of (4) is the determinant of the impedance matrix, which is nonzero. Therefore, the two linear equations (2), relating the four variables V_1, I_1, V_2, I_2 of the two-port, are independent. Two $U\overline{RC}$s are called *commensurate* if they have equal time constants. Pairs of commensurate $U\overline{RC}$s have been used widely in past works.

III. Grounded $U\overline{RC}$ Connected to Memoryless Linear Network (MLLN)

Our goal is to realize a rational transfer function (from a scalar input U to a scalar output Y) using only one grounded $U\overline{RC}$ and no other dynamic element. Therefore, our network must consist of the single $U\overline{RC}$ and an *MLLN*. The *MLLN* could be composed of ideal amplifiers, transconductors, gyrators, resistors, etc. The general scheme is shown in Fig. 2.

The *MLLN* must impose two linear relations between the variables V_1, I_1, V_2, I_2, U of the network in addition to the constitutional relations of the $U\overline{RC}$ given by (2)

$$\begin{aligned} k_{1,1}V_1 + k_{1,2}V_2 + k_{1,3}I_1 + k_{1,4}I_2 + \mu_1 U &= 0 \\ k_{2,1}V_1 + k_{2,2}V_2 + k_{2,3}I_1 + k_{2,4}I_2 + \mu_2 U &= 0. \end{aligned} \tag{5}$$

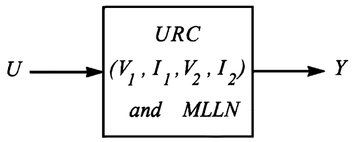

Fig. 2. $\boldsymbol{U\overline{RC}}$ connected to *MLLN*.

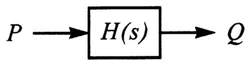

Fig. 3. $\boldsymbol{P}$–$\boldsymbol{Q}$ input–output relation of the $\boldsymbol{U\overline{RC}}$.

The *MLLN* must realize a relation involving the output as well

$$Y = k_{3,1}V_1 + k_{3,2}V_2 + k_{3,3}I_1 + k_{3,4}I_2 + \mu_3 U. \tag{6}$$

We say that the total network is *well defined* if the Kirchoffs equations have a unique algebraic solution for $V_1,\ I_1,\ V_2,\ I_2$, or equivalently, the set of *the four linear equations (2) and (5) is linearly independent with respect to* $V_1,\ I_1,\ V_2,\ I_2$ (for every U and $s \neq 0$).

Now, we assume that the total network is well defined. This implies that the two equations (5) are linearly independent and so the matrix

$$\begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\ k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} \end{bmatrix}$$

must have a nonzero 2 × 2 determinant. This implies that the pair of equations (5) can be solved for two of the variables in the set $\{V_1,\ I_1,\ V_2,\ I_2\}$. Let P_1 and P_2 be two such variables and let Q_1 and Q_2 be the remaining ones. In this case, (5) can be written equivalently as

$$\begin{aligned} P_1 &= a_1 Q_1 + a_2 Q_2 + b_1 U \\ P_2 &= a_3 Q_1 + a_4 Q_2 + b_2 U \end{aligned} \tag{7}$$

for some appropriate constants $a_1,\ \ldots,\ a_4$ and $b_1,\ b_2$. Replacing (7) into (6), we get (8) for some $c_1,\ c_2,\ d$

$$Y = c_1 Q_1 + c_2 Q_2 + dU. \tag{8}$$

The $U\overline{RC}$ element accepts all two-port descriptions [7]; one of them has to be given by (9) where $H(s)$ is the appropriate 2 × 2 matrix function of s

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = H(s) \cdot \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \tag{9}$$

It is worth mentioning here that all four entries of each matrix description of the $U\overline{RC}$ are irrational functions of s [7]. It is straight forward to verify that *it is impossible to realize a rational transfer function using a single* $U\overline{RC}$ *without feedback.*

Setting $P = [P_1,\ P_2]^T$ and $Q = [Q_1,\ Q_2]^T$, one can think of the $U\overline{RC}$ as a dynamical system, with the 2 × 2 transfer function matrix $H(s)$, whose input is P and its output is Q as shown in Fig. 3.

We define the matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

and the vectors $b = [b_1,\ b_2]^T$, and $c = [c_1,\ c_2]$. Then, (7)–(9) become $P = AQ + bU$, $Y = cQ + dU$ and $Q = H(s)P$, respectively, and are captured in the block diagram of Fig. 4.

The $U\overline{RC}$ corresponds to the central block $H(s)$, and equations (7) and (8) correspond to the feedback loop and the output part respectively. It is concluded that a well-defined network consisting of a $U\overline{RC}$ and a *MLLN* (with one input and one output) always admits the system representation of Fig. 4.

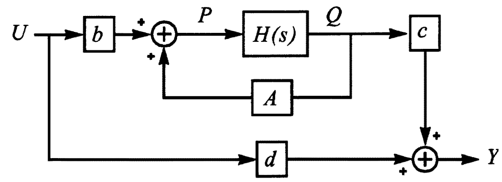

Fig. 4. Feedback representation of the total network.

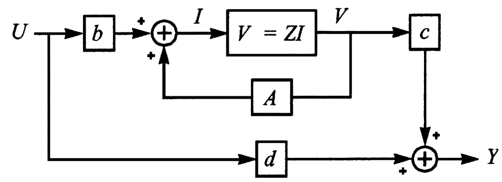

Fig. 5. Case of $\boldsymbol{P} = \boldsymbol{I}$ and $\boldsymbol{Q} = \boldsymbol{V}$.

Note that for every "renaming" of the variables $\{V_1,\ I_1,\ V_2,\ I_2\}$, we have a different transfer function $H(s)$. None of the matrices $H(s)$ has any entry that is a rational function of s [7]. On the other hand, there are some choices of the matrices A, b, c and d that result to an input–output rational transfer function $G(s)$ of the system in Fig. 4

$$G(s) = \frac{Y(s)}{U(s)}. \tag{10}$$

The transfer function $G(s)$ is given by expression (11) where E is the 2 × 2 identity matrix

$$G(s) = c(E - H(s)A)^{-1}H(s)b + d. \tag{11}$$

A. Implementation Using Transconductors

The voltage-dependent current source or transconductor is in general the easiest to implement and the most common type of dependent linear source. Here, we assume that matrix A is implemented by a number of transconductors. In this case we have: $P = I = [I_1,\ I_2]^T$ and $Q = V = [V_1,\ V_2]^T$. The input U is a current and we choose the output Y to be a current as well since the integrators may form cascades. Then, b and d must be dimensionless and c must be a transconductance. With these choices the general system of Fig. 4 is reduced to that of Fig. 5.

Matrix Z is the impedance matrix of the $U\overline{RC}$ given by (2) and (3). By replacing b, Z, A, c and d into (11) we have the following expression for the transfer function $G(s)$:

$$G(s) = \frac{\eta_0 Z_0 + \eta_m Z_m + \eta_s \frac{R}{Cs}}{1 - \rho_0 Z_0 - \rho_m Z_m + \rho_s \frac{R}{Cs}} + d \tag{12}$$

where

$$\begin{aligned} \eta_0 &= b_1 c_1 + b_2 c_2 \\ \eta_m &= b_1 c_2 + b_2 c_1 \\ \eta_s &= -a_4 b_1 c_1 + a_2 b_2 c_1 + a_3 b_1 c_2 - a_1 b_2 c_2 \\ \rho_0 &= a_1 + a_4 \\ \rho_m &= a_2 + a_3 \\ \rho_s &= a_1 a_4 - a_2 a_3. \end{aligned}$$

In general, $G(s)$ is an irrational function of s as are Z_0 and Z_m. A case in which $G(s)$ becomes rational, and more specifically a one-pole transfer function, is when the coefficients of Z_0 and Z_m are zero, i.e., when

$$b_1 c_1 + b_2 c_2 = 0 \qquad b_1 c_2 + b_2 c_1 = 0 \tag{13}$$

and

$$a_1 + a_4 = 0 \qquad a_2 + a_3 = 0. \tag{14}$$

The system of equations (13) can be written in matrix form

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{15}$$

Since the vector $b = [b_1, b_2]^T$ (and the vector $c = [c_1, c_2]^T$) cannot be zero (except in pathological cases), the determinant of the system (15) must be zero, so $c_2 = \pm c_1$. Therefore there are two possible pairs (up to a multiplicative factor) of b and c that satisfy (13), i.e.

$$\begin{aligned} &\text{i)} \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [c_1, c_2] = \gamma[1, -1] \\ &\text{ii)} \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad [c_1, c_2] = \gamma[1, 1] \end{aligned} \tag{16}$$

for some dimensionless constant β and some transconductance γ. Equations (14) imply that $a_3 = -a_2$ and $a_4 = -a_1$ which combined with equations (16) give the transfer functions

$$\text{i)} \qquad G(s) = \frac{2(a_1 + a_2)\beta\gamma R}{cs + (a_2^2 - a_1^2)R} \tag{17}$$

$$\text{ii)} \qquad G(s) = \frac{2(a_1 - a_2)\beta\gamma R}{cs + (a_2^2 - a_1^2)R}. \tag{18}$$

From (17) and (18), we get a lossless integrator when $a_1 = a_2 = -a_3 = -a_4$, and when $a_1 = -a_2 = a_3 = -a_4$, respectively. These two cases correspond to Integrator (I) and Integrator (II) topologies examined in the following.

IV. Grounded-$U\overline{RC}$ Integrator (I)

Here, we examine the lossless integrator corresponding to Case i), (16), (17). We set

$$\left.\begin{aligned} A = \alpha \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad & b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ c = \gamma[1, -1] \qquad & d = 0 \end{aligned}\right\}. \tag{19}$$

With these choices, the abstract network in Fig. 5 is realized by that of Fig. 6. The input current flows *symmetrically* into the two ports of the $U\overline{RC}$. The feedback loop is realized by the two *internal* voltage-dependent current sources that drive *opposite* currents into the two ports.

The transfer function of the network can be obtained by replacing the values of the parameters A, b, c, d into (12) or (17). This results in the following expression:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4\alpha\gamma R}{C} \cdot \frac{1}{s}. \tag{20}$$

Since α and γ are transconductances, the products $K_a = \alpha R$ and $K_\gamma = \gamma R$ are dimensionless quantities representing an *internal* and an *output* amplification factor, respectively.

Further discussion on the operation principle of Integrator (I) and how this is related to the separation of the *common/differential* modes of the $U\overline{RC}$ can be found in [8].

V. Grounded-$U\overline{RC}$ Integrator (II)

Here, we derive the lossless integrator corresponding to Case ii), (16) and (18). We set

$$\left.\begin{aligned} A = \alpha \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad & b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ c = \gamma[1, 1] \qquad & d = 0 \end{aligned}\right\}. \tag{21}$$

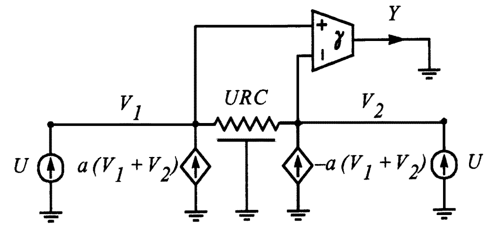

Fig. 6. Integrator (I).

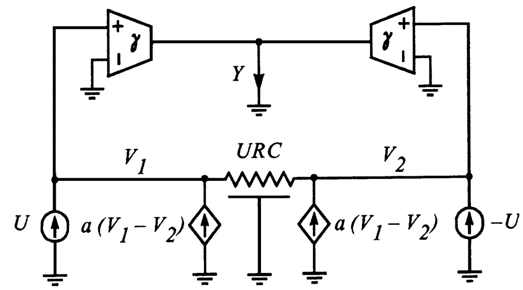

Fig. 7. Integrator (II).

The abstract network in Fig. 5 is realized by that of Fig. 7. In this case, the input is current U that flows *antisymmetrically* into the two ports of the $U\overline{RC}$. The constant β was again set to one. The feedback loop is realized by the two "internal" voltage dependent current sources that drive the two ports *symmetrically*.

The transfer function of the network is the same with that of Integrator (I) given by expression (20). A discussion on the operation principle of Integrator (II) and its relation to the *common/differential* modes of the $U\overline{RC}$ can also be found in [8].

VI. Conclusion

This brief has shown that integrators can be built out of a single grounded $U\overline{RC}$ (uniformly distributed RC element). This is in contrast to all previously proposed integrator topologies based on distributed elements, which required pairs of exactly matched (commensurate) $U\overline{RC}$s or other complicated distributed structures. Here, the dynamic behavior of memoryless linear feedback loops around a grounded $U\overline{RC}$ has been studied and has led to two novel integrator topologies. An alternative viewpoint and derivation can be found elsewhere [8]. It can be shown that there are four additional classes of lossy integrators that result from the presented feedback approach and use only transconductors and one grounded $U\overline{RC}$. There are many more classes of integrators and lossy integrators that use other types of dependent sources; these can be derived using a systematic deduction along the lines presented here.

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Design and Analysis of Nonlinear Control for Uncertain Linear Systems

Xinkai Chen and Chun-Yi Su

***Abstract*—By using the input–output information, the problem of robust output tracking control is addressed for linear dynamical systems with arbitrary relative degrees. The considered systems are confined to minimum phase systems with unknown parameters, and unmatched disturbances composed of a bounded part and a class of unmodeled dynamics. The *a priori* knowledge concerning the disturbance bounds is unknown. The development of the nonlinear robust controller involves three steps. First, a special signal is generated, which can be thought of as an estimate of a filter of the input signal. Second, the derivatives up to a certain order of this special signal are derived. Third, the output tracking control input is synthesized by using the derivatives of the special signal. In the above process, the upper bounds of the disturbances are adaptively updated on-line. The proposed control law ensures the uniform boundedness of all the signals in the closed-loop system and achieves the output tracking to within a desired precision. The effectiveness of the proposed method is demonstrated through simulation.**

***Index Terms*—Input–output information, minimum phase systems, output tracking, relative degree, robust control, unmatched uncertainty.**

I. Introduction

In robust output tracking control, a central problem is to design a feedback control for a plant such that the output of the plant can asymptotically track a class of reference signals and reject a class of disturbances while maintaining closed-loop stability. For the class of linear systems, the solvability of the output tracking problem was thoroughly studied in [3], [4], and [7]–[11]. However, the system disturbances are generally assumed to be either constant or bounded. For minimum phase systems with unknown parameters and bounded disturbances, several typical adaptive methods achieving output tracking were suggested in [5], [6].

For systems with uncertainties, variable structure control has been investigated in robust control literature because of its effective performances [12], [13], [15]. However, in this kind of approach, the system uncertainties or disturbances are still assumed both bounded and matched. Also, the results are restricted to minimum phase dynamical systems with relative degree one. The proposed formulations cannot cope with systems of higher relative degrees, and cannot deal with unmatched disturbances or uncertainties. In the variable structure control, the unmatched disturbances become part of the equivalent control and must be estimated for the construction of the equivalent control.

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For systems with unknown parameters and unmatched disturbances, an interesting robust approach is developed in [14] based on state-space techniques, where the input–output information and the *a priori* knowledge concerning the disturbance bounds are used. The overall system can be ensured to be globally uniformly ultimately bounded (GUUB) which can be made arbitrarily close to exponential stability if the control energy permits. However, the perfect *a priori* knowledge concerning the disturbance bounds may not be easily obtained in practice.

This brief demonstrates the design of a nonlinear output tracking controller for systems with both unknown parameters and unmatched disturbances. The unmatched disturbances are composed of a bounded part and a class of unmodeled dynamics. The perfect *a priori* knowledge concerning the disturbance bounds is not required. The disturbance bounds are adaptively updated online. The considered systems may have higher relative degrees. The proposed formulation is inspired by the "nonlinear differentiator" proposed in [1], and [2], which is motivated by the variable structure control and adaptive control methods. The design procedure in this brief can be summarized as three steps. First, a special signal is generated, which can be thought of as an estimate of a filter of the input signal. Second, the derivatives up to a certain order of this special signal are derived, where a backstepping idea [4] is used. Third, the output tracking control input is synthesized by using the derivatives of the special signal. The proposed nonlinear control law ensures the uniform boundedness of all the signals in the closed-loop system and achieves output tracking within a desired precision. The effectiveness of the proposed method is demonstrated through simulation.

This brief is organized as follows. Section II gives the problem formulation. In Section III, firstly, a special signal (which can be thought of as an estimate of a filter of the input signal) is generated. Secondly, the derivatives up to a certain order of the special signal are derived. Finally, the output tracking control input is determined, and the stability of the closed-loop system is analyzed. Section IV gives a design example to illustrate the proposed formulation. Section V provides conclusions.

II. Problem Statement

Consider an uncertain system of the form

$$a(s)y(t) = b(s)u(t) + v(t) \tag{1}$$

where s denotes the differential operator; $u(t)$ and $y(t)$ are scalar input and output, respectively; $v(t)$ is an unknown signal composed of model uncertainties, nonlinearities and disturbances, etc.; $a(s)$ and $b(s)$ are described by

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{2}$$
$$b(s) = b_r s^{n-r} + b_{r+1}s^{n-r-1} + \cdots + b_{n-1}s + b_n. \tag{3}$$

It can be easily seen that $v(t)$ is an unmatched unknown signal. For simplicity, the signal $v(t)$ is called the "disturbance" of the system. It is assumed that the initial time is t_0.

The following assumptions are made.

(A1) $b(s)$ is a Hurwitz polynomial. $a(s)$ and $b(s)$ are coprime.

(A2) The indexes n and r are known. $b_r \neq 0$ and the sign of it is known. Without loss of generality, it is assumed $b_r > 0$.

(A3) The parameters in $a(s)$ and $b(s)$ are unknown constants but they are bounded in known compact sets. More specifically, there are known constants $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_j$ and $\overline{b}_j$ such that for $1 \leq i \leq n$ and $r \leq j \leq n$

$$\underline{a}_i \leq a_i \leq \overline{a}_i \quad \underline{b}_j \leq b_j \leq \overline{b}_j \tag{4}$$

where $\underline{b}_r > 0$.