Time-Near-Optimal Longitudinal Control for Quadrotor UAVs

Dimitris Nikitas
Dept. of Electrical and Computer Engineering
National Technical University of Athens, Greece
el15006@central.ntua.gr

Konstantinos Papafotis
Dept. of Electrical and Computer Engineering
National Technical University of Athens, Greece
kpapafotis@central.ntua.gr

Paul P. Sotiriadis
Dept. of Electrical and Computer Engineering
National Technical University of Athens, Greece
pps@ieee.org

2021 10th International Conference on Modern Circuits and Systems Technologies (MOCAST) | 978-1-6654-1847-8/21/$31.00 ©2021 IEEE | DOI: 10.1109/MOCAST52088.2021.9493385

***Abstract*—In this work we propose a near-optimal Bang-Bang controller for the longitudinal motion of a quadrotor UAV which minimizes the active flight time. The proposed controller is closed loop, has simple cascaded structure and can be used in online navigation since it has minimum computational requirements compared to other approaches in the literature. It is evaluated via numerical simulation for a standard quadrotor nonlinear model.**

***Index Terms*—quadrotor, optimal control, least time, bang-bang**

I. Introduction

Quadrotors and UAVs (Unmanned Aerial Vehicles) are receiving considerable interest from both the commercial and the scientific field, in the last decade due to their cost saving technology and the variety of possible applications. The quadrotor unmanned aerial vehicle is a great platform for control systems research as its nonlinear nature and under-actuated configuration make it ideal to synthesize and analyze control algorithms [1].

Currently, one of the main drawbacks of UAVs is the limited flying time. Quadrotors are powered by high-performance lithium batteries that usually last for about 20 to 30 minutes. Environmental dynamic disturbances increase power consumption and flight times are shortened. Short periods of time matter. Therefore, time efficiency is a very important metric for a control strategy. Considering that model is strongly coupled and nonlinear, solving least time problem using Pontryagin's Minimum Principle [2] displays deterring computational cost.

In the wider category of time optimal control strategies, optimizations are proposed in a numerical approach by minimizing criteria functions, like in [3]. These strategies suffer from high computational complexity since they require heavy optimization techniques like Non Linear Programming (NLP). These computations are not viable for an embedded micro-controller. D'Andrea et al [4] have solved the problem of closed loop controller for a simplified 2D model, by finding a priori the control's switching times numerically. Even though they solve the problem using analytical tools, it is stated that results can be only regarded as benchmarks due to the high computational cost.

In this work, we tackle the 2D planar motion time optimal problem for rest to rest transition, developing the idea of Cascaded Bang-Bang Control. Our control scheme is closed loop, ensuring robustness and efficiency for a system with limited computational capabilities. Our controller's performance is compared to that of a traditional, yet prevalent controller.

The paper is structured as it follows. In Section II, a mathematical model for UAVs is being presented and we touch upon some necessary but well-known analytical results, for Bang-Bang control for Double Integrator systems. Section III presents the core of our control algorithm. The efficiency of our algorithm is underlined through simulation results in Section IV.

II. Preliminaries

A. Mathematical Model

Quadrotors are modelled as symmetric 3D rigid bodies. Propellers produce vertical thrust over the aircraft's attitude. Translation motion is expressed with respect to inertia frame. Angle representation for rotational motion is done using Euler Angles (roll ϕ, pitch θ, yaw ψ) and the corresponding Rotation Matrices.

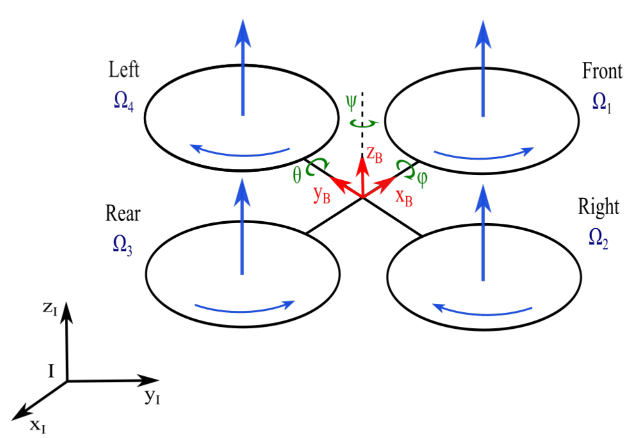

Fig. 1. Quadrotor simplified model. Propellers' velocities Ω_i are expressed in rpm.

The standard quadrotor model [5], [6] is

$$\begin{aligned} m\ddot{r} &= -mge_z + Rf_T e_z \\ \dot{\omega} &= (I\omega)\times\omega + I\tau \end{aligned} \tag{1}$$

where

m : mass of quadrotor
g : gravity acceleration
e_z : 3×1 frame vertical vector $[0\ 0\ 1]^T$
r : 3×1 position vector $[x\ y\ z]^T$
ω : 3×1 angle velocity vector $[\dot{\phi}\ \dot{\theta}\ \dot{\psi}]^T$
I : 3×3 moment of inertia matrix
f_T : vertical thrust
τ : 3×1 torque vector $[\tau_\phi\ \tau_\theta\ \tau_\psi]^T$
R : 3×3 rotation matrix transforming body to inertia frame

Rotation matrix is defined as

$$R = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix}$$

where $c_{(\cdot)} = \cos(\cdot), s_{(\cdot)} = \sin(\cdot)$. For a symmetric aircraft $I = \text{diag}(I_x, I_y, I_z)$. Thrust f_T and torque τ are given in the following way

$$\begin{bmatrix} f_T \\ \tau \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ 0 & -bl & 0 & bl \\ -bl & 0 & bl & 0 \\ -d & d & -d & d \end{bmatrix} \begin{bmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{bmatrix} \tag{2}$$

where

b : thrust factor
d : drag factor
l : propeller's center to quadrotor's center of mass distance
Ω_i : $i-$propeller's angular velocity

Since coefficient matrix is invertible, desired generalized force can be translated to desired propellers' velocities.

B. Review of Bang-Bang Controller for Double Integrator

The double integrator is the simplest linear non-trivial second order model and allows us to extract very important analytical results.

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= u \end{aligned} \tag{3}$$

We are then faced with the challenge to find the control strategy that minimizes the time needed for the system to transit from a starting point to a final point. So the cost function is $J = t_f$. When input is bounded (i.e. $|u| \le M$) representation is more realistic. The solution is called Bang-Bang control because control switches between extreme inputs, $u = \pm M$. The analysis is standard and thus, can be found in books of linear Optimal Control like Naidu's [7] and Athans's [2].

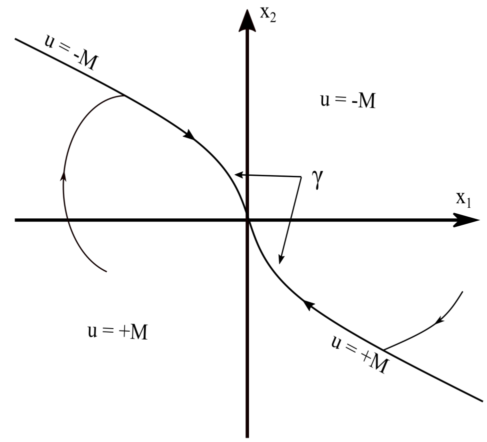

Fig. 2. Switching curve $\gamma = 0$ and optimal trajectories for several initial states.

For the double integrator there might be no or one switch. Let γ be the switching function

$$\gamma(x_1(t), x_2(t)) \triangleq x_1 + \frac{1}{2M}x_2|x_2| \tag{4}$$

assuming origin is the desired final state considering and M is the input bound. The closed loop optimal control is

$$u = f^*(x_1, x_2) \triangleq \begin{cases} -M\text{sgn}(\gamma) & \text{for } \gamma \neq 0 \\ M\text{sgn}(x_1) & \text{for } \gamma = 0 \end{cases} \tag{5}$$

sgn denotes the sign function. Function f^* denotes the Bang-Bang control solution for the double integrator and will be used repeatedly in this paper. The above closed loop control drives the system from initial conditions to the origin. By setting $x_1' = x_1 - x_{1d},\ x_2' = x_2$ and using the above control for the new states, the system can be driven to any desired point x_{1d} at rest. The $\gamma = 0$ curve is called switching curve because as soon as the system crosses the curve the input should be switched.

III. Proposed Near-Optimal Controller

In this section we investigate the planar motion problem in the least time. To begin with, we assume an initial point $(x_0, y_0) = (0, 0)$ with zero starting velocity and a desired final point (x_d, y_d) to be reached at rest.

Planar motion is equivalent to longitudinal motion in one axe as soon as drone is heading towards the target. Therefore, before translation motion begins, the drone has to adjust its orientation to be aligned with the target point. This procedure is quite standard. Assuming that initial yaw is zero, the desired fixed yaw ψ_d can be calculated as

$$\psi_d = \text{atan2}(y_d, x_d) \tag{6}$$

To maintain aircraft at a specific initial height z_0, thrust f_T must satisfy $\ddot{z} = 0$. Thus

$$\begin{aligned} f_T &= \frac{1}{c_\phi c_\theta}\left(mg - k_1(z - z_0) - k_2\dot{z}\right),\ k_1, k_2 > 0 \\ &\approx \frac{mg}{c_\phi c_\theta} \end{aligned} \tag{7}$$

In order to not lose controllability over z axe we should ensure that $|\phi| \leq \phi_{max} < \pi/2, |\theta| \leq \theta_{max} < \pi/2$, which is a common constraint for non-maneuvering, general purpose drones. Position dynamics, after choosing applied thrust and desired yaw, become

$$\begin{aligned} m\ddot{x} &= (c_\phi s_\theta c_{\psi_d} + s_\phi s_{\psi_d})\frac{mg}{c_\phi c_\theta} \\ m\ddot{y} &= (c_\phi s_\theta s_{\psi_d} - s_\phi c_{\psi_d})\frac{mg}{c_\phi c_\theta} \\ m\ddot{z} &= 0 \end{aligned} \tag{8}$$

To simplify dynamics we apply the following transformation by defining the auxiliary position state $r' = Tr$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi_d & \sin\psi_d & 0 \\ -\sin\psi_d & \cos\psi_d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

This transformation aligns x' direction with target point, in a way that only longitudinal motion is required. Double differentiating r' gives the following simplified structure

$$\begin{aligned} \ddot{x}' &= g\tan\theta \\ \ddot{y}' &= -g\frac{\tan\phi}{\cos\theta} \\ \ddot{z}' &= 0 \end{aligned} \tag{10}$$

Since lateral motion is unnecessary we can let $\phi = 0$. Moreover, a constraint bound M in torque τ_θ arises from the fact that propellers can produce finite angular velocity and the internal constraint for non-negative solution in (2). We obtain the longitudinal model of reference

$$\begin{aligned} \ddot{x}' &= g\tan\theta & |\theta| &\leq \theta_{max} \\ \ddot{\theta} &= [(I\omega)\times\omega]_2 + I_y\tau_\theta & |\tau_\theta| &\leq M \end{aligned} \tag{11}$$

where $[\cdot]_2$ denotes second element. Apparently, $[(I\omega)\times\omega]_2 = 0$ due to constant angle $\psi = \psi_d$. By setting the auxiliary variables $u = g\tan\theta$ and $u_\theta = I_y\tau_\theta$ we get

$$\begin{aligned} \ddot{x}' &= u \\ |u| &\leq g\tan\theta_{max} \end{aligned} \tag{12}$$

where

$$u = g\tan\theta \tag{13}$$

and

$$\begin{aligned} \ddot{\theta} &= u_\theta \\ |u_\theta| &\leq MI_y^{-1} \end{aligned} \tag{14}$$

To not overload equations we will set x' as x which is different from initial position vector element. Considering (11) in a back-stepping control scope, $g\tan(\theta)$ is the virtual control for $(x,\dot{x})$ system. Thus, the longitudinal subsystem is virtually divided into a pure double integrator and a double integrator with delayed control, like mentioned above. Dynamics (14) simulate the dynamics of virtual control u. If we demand $(\theta,\dot{\theta})$ system (14) to generate only extremum angles $\theta_d = \pm\theta_{max}$, so that $u = \pm g\tan\theta_{max}$, then control strategy (13) can be considered close to Bang-Bang for $(x,\dot{x})$ subsystem. To accelerate achieving the desired angle θ_d, u_θ must be selected to be Bang-Bang as well. Considering $(x,\dot{x})$ subsystem as a double integrator, it is expected to have, at most, one switch. Figure 3 offers a block diagram of the strategy. All in all, a Bang-Bang control driven double integrator (14) generates appropriate Bang-Bang signal (13) for another double integrator like system (12).

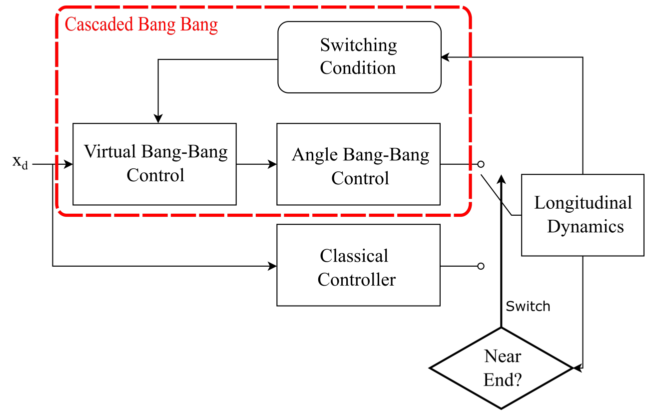

Fig. 3. Proposed controller block diagram structure.

For this scheme to work we should guarantee that torque τ_θ can take values $\geq 5I_y^{-1}g\tan\theta_{max}$ so that inner loop $(\theta,\dot{\theta})$ is at least 5 to 10 times faster in comparison with outer loop $(x,\dot{x})$ which is something very usual in UAVs. This is essential because the time needed for $\theta = \pm\theta_{max}$ signal to be generated would be negligible for the slower dynamics of $(x,\dot{x})$ system.

In other words, a real Bang-Bang control generates another virtual Bang-Bang control. Thus, it's a cascaded Bang-Bang control approach. So, there are two subsystems with each of them separately working in the means of least time. However, there is no ensuring that optimality in the subsystems entail optimality for the full longitudinal system. Therefore, this is a sub-optimal solution. The virtual Bang-Bang controller is defined as

$$u = u_x = f^*(x - x_d, \dot{x}) \tag{15}$$

and the switching function, according to (4), is as it follows

$$\gamma_x = x - x_d + \frac{1}{2g\tan\theta_{max}}\dot{x}|\dot{x}| \tag{16}$$

Thus, it is necessary that the desired angle θ_d follows virtual input u_x as

$$\theta_d = \text{sgn}(u_x)\theta_{max} \tag{17}$$

Inner loop Bang-Bang controller is defined as

$$u_\theta = f^*(\theta - \theta_d, \dot{\theta}) \tag{18}$$

with switching function

$$\gamma_\theta = \theta - \theta_d + \frac{1}{2MI_y^{-1}}\dot{\theta}|\dot{\theta}| \tag{19}$$

Since the system is not a perfect double integrator it will probably not stop at the desired point with zero velocity. To avoid this imperfection, it is legitimate to switch to a slower

TABLE I
QUADCOPTER MECHANICAL PARAMETERS

Symbol	Quantity	Value
m	mass	$1\ [kg]$
I_x	moment of inertia about x	$8.1\ 10^{-3}\ [kg\ m^2]$
I_y	moment of inertia about y	$8.1\ 10^{-3}\ [kg\ m^2]$
I_z	moment of inertia about z	$14.2\ 10^{-3}\ [kg\ m^2]$
l	center to propeller distance	$0.24\ [m]$
b	thrust factor	$54.2\ 10^{-6}\ [kg\ m]$
d	drag factor	$1.1\ 10^{-6}\ [kg\ m^2]$

linear or nonlinear controller [8] near the desired point for sufficient stabilization.

IV. SIMULATION

The proposed optimal controller is evaluated using numerical simulation in MATLAB environment. For increased accuracy, a first order model with time constant $\tau = 0.1$ is considered for propellers' response. Typical mechanical parameters for a quadrotor UAV, that were assumed in the simulation, are listed in Table I.

In order to evaluate controller's efficiency, we compare performance between a leading quadrotor controller [8], [9] and our near-optimal controller. We assume $(x_0, y_0) = (0, 0)$ and $(x_d, y_d) = (300, 0)$. A reasonable choice for θ_{max} is $\pi/4$, while it can be calculated that $M = 7400$. Furthermore, the final times of the respective controllers are compared to true optimal time [2].

As we can see in Figure 4 our algorithm's performance approaches ideal execution time with success. Near the end point we notice an abnormality. This is because a nonlinear controller is enabled for stabilization. For better insight of the Cascaded Bang-Bang controller's operation, angle control, virtual control and γ_x function response in time are shown in Figure 5. It can be observed that when $\gamma_x = 0$ virtual control switches as desired.

At true optimal time, position error is just ~ 0.1, while the traditional controller needs approximately 4 more seconds to settle. Our controller displays 20% final time improvement in this numerical experiment.

V. CONCLUSION

In this research, a new optimal control scheme for UAVs is proposed, the Cascaded Bang-Bang control, which supersedes cascaded PID or similar structured nonlinear techniques as illustrated by numerical example.

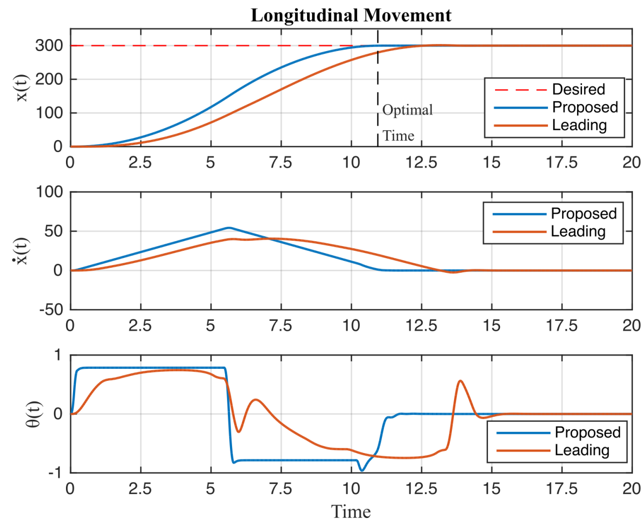

Fig. 4. Longitudinal Control: Position, velocity and angle θ. Quantities are expressed in SI units. Comparison between a leading controller and our controller with true optimal time.

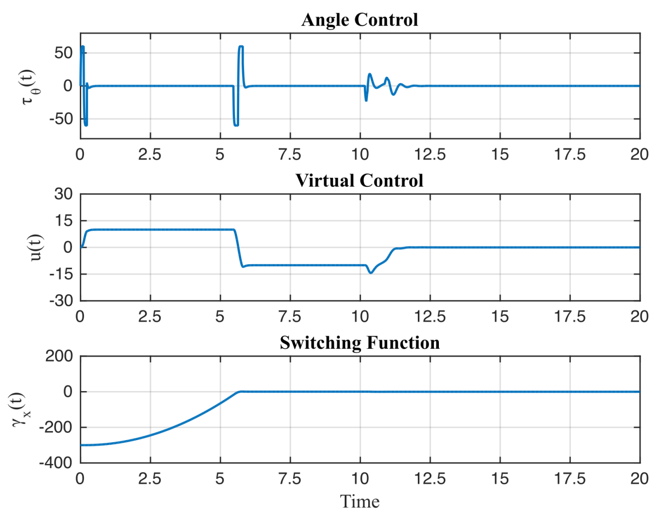

Fig. 5. Longitudinal Control: Angle and Virtual Bang-Bang controller's signals and switching function γ_x. Quantities are auxiliary and thus units are not meaningful.

REFERENCES

[1] A. Zulu and S. John, "A review of control algorithms for autonomous quadrotors," *Open Journal of Applied Sciences*, vol. 4, pp. 547–556, 2014.
[2] M. Athans and P. Falb, *Optimal Control: An Introduction to the Theory and Its Applications*. New York: Dover Publications, 2007.
[3] K. Eliker, G. Zhang, S. Grouni, and W. Zhang, "An optimization problem for quadcopter reference flight trajectory generation," *Journal of Advanced Transportation*, vol. 2018, 2018.
[4] M. Hehn, R. Ritz, and R. D'Andrea, "Performance benchmarking of quadrotor systems using time-optimal control," *Autonomous Robots*, vol. 33, p. 69–88, 2012.
[5] S. Bouabdallah, "Design and control of quadrotors with application to autonomous flying," Ph.D. dissertation, EPFL, Faculty of Sciences and Enginnering, Autonomous Systems Laboratory, Lausanne, Switzerland, 2007.
[6] P. Ponds and C. R. Mahony, "Modelling and control of a large quadrotor robot," *Control Engineer Practise*, vol. 18, pp. 691–699, 2010.
[7] Naidu and D. Subbaram, *Optimal Control Systems (Electrical Engineering Series)*. University of California: CRC Press, 2003.
[8] T. Madani and A. Benallegue, "Backstepping control for a quadrotor helicopter," in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, Beijing, China, 2006.
[9] S. Bouabdallah, P. Murrieri, and R. Siegwart, "Design and control of an indoor micro quadrotor," in *IEEE International Conference on Robotics and Automation, 2004*, New Orleans, LA, USA, 2004.