Derivation of the Transfer Functions of 1-bit Multi-Step Look-Ahead ΣΔ Modulators Using System Identification Methods

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***Abstract*—A method to calculate the transfer functions of Multi-Step Look-Ahead (MSLA) ΣΔ modulators is presented. MSLA ΣΔ modulators exhibit better noise shaping characteristics and stability than conventional ones. They are comprised of a number of conventional ΣΔ modulators in parallel, sharing a multi-input 1-bit output quantizer. MSLA modulators are highly nonlinear systems due to the multi-input quantizer. Modeling of the quantizer using conventional linearization methods does not give satisfactory results. Therefore, this work applies system identification methods to derive the linearized MSLA modulator system transfer functions. More specifically the Vector Fitting algorithm is used for the linearization of a number of MSLA modulators. The obtained transfer functions are in very good agreement with simulation results, showcasing the effectiveness of the applied methods.**

***Index Terms*—Sigma-delta, noise shaping, 1-bit quantization, modulator, all-digital, optimization algorithm, look-ahead, system identification, nonlinear system, Vector Fitting algorithm**

I. Introduction

Many electronic systems have a ΣΔ modulator as an integral component. These systems range from data converters to fractional-N PLLs, all-digital transmitters, class-D power amplifiers and digital bitstream encoders. ΣΔ modulators exploit oversampling, i.e. sampling at a much higher rate than the Nyquist rate, in order to accurately represent a signal using just 1 or few bits per sample [1]. The higher the oversampling ratio (OSR), the better the accuracy. The system of a ΣΔ is comprised of a feedback loop and a loop filter as shown in Fig. 1.

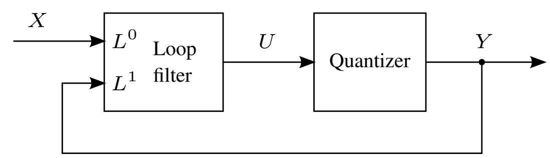

Fig. 1. System diagram of a ΣΔ modulator.

In most designs there is only a single-input filter L, i.e. not a two-input one as shown in Fig. 1, and the difference of x and y, $x - y$, is fed to the loop filter. The loop filter may be low-pass, band-pass or high-pass depending on the application and defines the frequency range in which the quantization error between the input and the output is minimized. The loop bandwidth and the out-of-band gain are limited by stability considerations [1]. Furthermore, stability requirements also pose restrictions to the allowable input signal range.

The quantizer resolution of ΣΔ modulators also affects their stability. Single-bit modulators are more susceptible to instability than multi-bit ones. However, multi-bit quantizers require multi-bit DACs for the reconstruction of the analog signals. Multi-bit DACs exhibit nonlinear behavior due to device mismatches, whereas 1-bit ones are inherently linear since there are only two signal levels and therefore only gain and offset errors. These errors do not degrade the output signal spectrum and can be easily eliminated using calibration methods. Moreover, many applications such as class-D power amplifiers and 1-bit bitstream encoders require 1-bit quantization.

Many works have dealt with the stability analysis of ΣΔ modulators. Parallel decomposition of high-order modulators, limit cycle analysis [2] and quasi-linear modeling using the describing function method [3] are some of the proposed techniques.

Multi-Step Look-Ahead (MSLA) ΣΔ modulators [4] have been proposed to improve upon the stability and noise shaping characteristics of conventional 1-bit ΣΔ modulators. They are highly nonlinear systems and thus their analysis using linear techniques like the ones used for conventional ΣΔ modulators is not straightforward. This work proposes a system identification approach using the Vector Fitting algorithm [5] for the derivation of a linearized MSLA modulator system description. This linearized system can then be used to analyze MSLA modulators using techniques applicable to conventional ΣΔ modulators.

The next section presents the basic system description of MSLA modulators. In section III the proposed method for the derivation of the MSLA modulator linearized transfer functions is presented, followed by supporting simulation results. Finally, section IV concludes the discussion.

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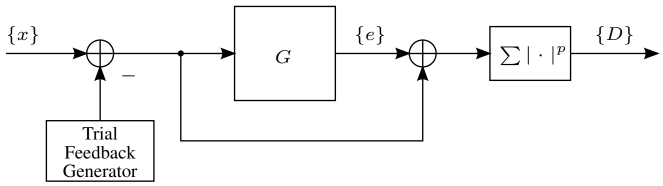

Fig. 2. The MSLA modulator as an optimization algorithm.

II. The MSLA Modulator System Description

MSLA modulators are a variation of conventional 1-bit ΣΔ modulators. Their main difference is that MSLA modulators take into account the current and k future output samples for the minimization of the quantization error in the pass-band of the loop filter, whereas conventional ΣΔ modulators only consider the current output. This results in better noise shaping characteristics, i.e. higher SNDR (signal to noise and distortion ratio), and improved stability over conventional ΣΔ modulators. For detailed simulation results and explanation of the underlying algorithms please see [4]. Only the key ideas required for the analysis presented in this work are covered here.

A. The MSLA Modulator as an Optimization Algorithm

In [4] it is shown that the output of the MSLA modulator is given by the optimization algorithm

$$y_n = \underset{v_0 \in \{\pm 1\}}{\arg\min} \left(\min_{\substack{v_1, v_2, \ldots, v_k \in \\ \{\pm 1\}}} \sum_{j=k-r}^{k} |x_{n+j} + e_{n+j} - v_j|^p \right). \tag{1}$$

This is illustrated in Fig. 2, where $D = \sum_{j=k-r}^{k} |x_{n+j} + e_{n+j} - v_j|^p$. Filter G is known as the *comparison filter* and it plays the same role as the loop filter of conventional ΣΔ modulators. It is related to the noise transfer function (NTF) of a conventional ΣΔ modulator by the equation

$$G(z) = \frac{1 - NTF(z)}{NTF(z)} = \frac{\sum_{i=1}^{\ell} b_i z^{-i}}{1 + \sum_{i=1}^{m} a_i z^{-i}}. \tag{2}$$

The output of the comparison filter at discrete time instant n is denoted e_n. The number of future output samples taken into account for the minimization of the quantization error, from now on *look-ahead steps*, is k and $r+1$ is the number of partial costs $S_{j,n}(v_0, v_1, \ldots, v_j) \equiv |x_{n+j} + e_{n+j} - v_j|^p$ used for the minimization. Parameter p denotes that the cost function is the p-norm distance of the optimizing variables vector $\mathbf{v} = (v_{k-r}, v_{k-r+1}, \ldots, v_k)$ from the vector incorporating the input and the comparison filter output samples $(x_{n+k-r} + e_{n+k-r}, x_{n+k-r+1} + e_{n+k-r+1}, \ldots, x_{n+k} + e_{n+k})$. The most common choice is $p = 2$ as it minimizes the total quantization error power yielding the best SNDR. However, a value of $p = 1$ leads to a more efficient hardware implementation [4] with a low SNDR cost.

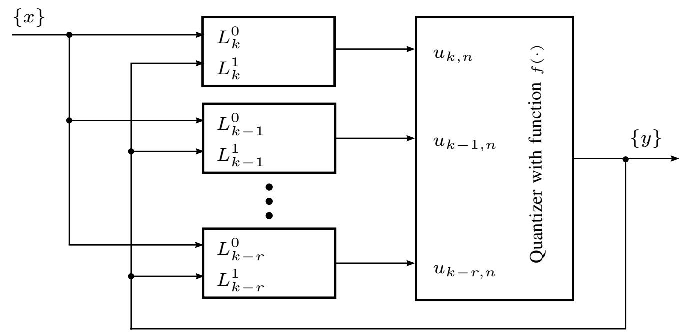

Fig. 3. The MSLA modulator efficient form system diagram.

B. MSLA Modulator Efficient Form

An equivalent system description of the MSLA modulator, avoiding the exponential algorithmic complexity associated with (1), is shown in Fig. 3 [4]. In this description the modulator is comprised of $r+1$ two-input filters and a $(r+1)$-input 1-bit output quantizer.

The transfer functions of the filters in Fig. 3 are given by

$$L_j^0(z) = \sum_{i=0}^{j+\ell-1} c_{j,i} z^{j-i} + G(z) \sum_{i=0}^{m-1} d_{j,i} z^{-i} \tag{3}$$

$$L_j^1(z) = -\sum_{i=j+1}^{j+\ell-1} c_{j,i} z^{j-i} - G(z) \sum_{i=0}^{m-1} d_{j,i} z^{-i} \tag{4}$$

with $k - r \leq j \leq k$. Coefficients $c_{j,i}$ and $d_{j,i}$ are derived from the comparison filter G coefficients b_i and a_i. The equations defining their values are thoroughly discussed in [4]. The filter outputs $u_{j,n}$, $k - r \leq j \leq k$ are given by the difference equation $u_{j,n} = \sum_{i=0}^{j} c_{j,i} x_{n+j-i} + \sum_{i=j+1}^{j+\ell-1} c_{j,i} (x_{n+j-i} - y_{n+j-i}) + \sum_{i=0}^{m-1} d_{j,i} e_{n-i}$. The filter output vector $\mathbf{u} = (u_{k-r,n}, u_{k-r+1,n}, \ldots, u_{k,n})$ is then fed to the $(r+1)$-input 1-bit quantizer.

The quantizer mapping function $f(\cdot)$ depends on the comparison filter G, the number of look-ahead steps k and the number of partial costs $r+1$. It is a time-invariant function, i.e. it does not depend on n. In [4] it is shown that

$$f(\mathbf{u}) = \underset{v_0 \in \{\pm 1\}}{\arg\min} \left(\min_{\substack{v_1, v_2, \ldots, v_k \in \\ \{\pm 1\}}} \sum_{j=k-r}^{k} \left| u_{j,n} - \sum_{i=0}^{j} c_{j,i} v_{j-i} \right|^p \right). \tag{5}$$

Therefore, the MSLA modulator output is equivalently given by $y_n = f(u_{k-r,n}, u_{k-r+1,n}, \ldots, u_{k,n})$. Consequently y_n is determined by the least p-norm distance of the quantizer input vector $\mathbf{u}$ from a set of points with coordinates $\sum_{i=0}^{j} c_{j,i} v_{j-i}$ in u_j axes, $k - r \leq j \leq k$. There is one point for each possible sequence $\{v\} = (v_0, v_1, \ldots, v_k)$, $v_i \in \{\pm 1\}$, resulting in a total of 2^{k+1} points in $(r+1)$-dimensional space.

For a more thorough analysis the reader is referred to [4]. Therein it is also shown that MSLA modulators achieve the same or better performance than other look-ahead techniques with comparable algorithmic complexity.

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III. The Linearized MSLA Modulator System

A simple and approximate approach to the linearization of the MSLA modulator system in Fig. 3 is to replace the multi-input quantizer with a gain K_j and a noise source N_j for each of its inputs. This has been done in [4] where the noise (NTF) and signal transfer functions (STF) are derived as $NTF_{\mathrm{MSLA}} \equiv Y/N_k|_{X=0} = 1/\left(1 - K_k L_k^1\right)$ and $STF_{\mathrm{MSLA}} \equiv Y/X|_{N_i=0} = \left(K_k L_k^0\right)/\left(1 - K_k L_k^1\right)$ respectively. Furthermore, K_k is calculated by minimizing the average power of the quantizer linear model error $y_n - K_k u_{k,n}$, i.e. $K_k = \langle y, u_k\rangle / \langle u_k, u_k\rangle = \langle f(\mathbf{u}), u_k\rangle/\sigma_{u_k}^2$, where $\langle a, b\rangle$ is defined either stochastically as $E[ab]$ or deterministically as the time average $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} a_n b_n$ of the sequences a_n and b_n. The value of K_k can be derived via simulation.

Notice that only a single filter, i.e. (L_k^0, L_k^1), is considered in order to derive the whole system transfer functions. This is an indication that this analysis fails to capture all the dynamics of the MSLA modulator. Next, a system identification approach is used to derive the NTF of the MSLA modulator. To avoid confusion, the term $\mathrm{NTF}_{\mathrm{MSLA}}$ will be used when referring to the NTF of the MSLA modulator.

A. The Vector Fitting Algorithm

The Vector Fitting algorithm [5] is a numerical technique that uses a rational model approximation to match the observed frequency response of a system (dataset). For the linearization of the MSLA modulator system in Fig. 3, it is convenient to model its behavior based exclusively on frequency response magnitude data derived from simulation. Our linearization method relies on deriving a magnitude-equivalent, minimum-phase system to the NTF of the MSLA modulator and then fitting the equivalnet system with a LTI model, using the Vector Fitting algorithm.

1) The Minimum-Phase Magnitude-Equivalent System: Let $|X(e^{j\omega})|$ denote the magnitude of $\mathrm{NTF}_{\mathrm{MSLA}}$. Let us define $x[n]$ so that $X(e^{j\omega}) = \mathcal{F}(x[n])$. For a LTI system it holds that [6]

$$S_y = |H(e^{j\omega})|^2 S_x \tag{6}$$

where S_y, S_x are the power spectral densities of the output and input signals respectively. Assuming the input to the noise transfer function to be a white, uniformly distributed and with unity power spectral density over the frequency range of interest random process, it follows from (6) that the magnitude of $\mathrm{NTF}_{\mathrm{MSLA}}$ is $|X(e^{j\omega})| = \sqrt{S_y}$.

The derivation of the minimum-phase equivalent system is based on the minimum-phase and all-pass deconvolution method described in [6]. $X(e^{j\omega})$ is a real-valued function of ω and consequently the complex cepstrum of $x[n]$ exists. This is the only condition on which the sequence $x[n]$ can be expressed as a minimum-phase and an all-pass component, i.e.

$$x[n] = x_{min}[n] * x_{ap}[n]. \tag{7}$$

Noting that $|X_{ap}(e^{j\omega})| = 1$, and taking the Fourier transform of (7), the minimum-phase component is equivalent to

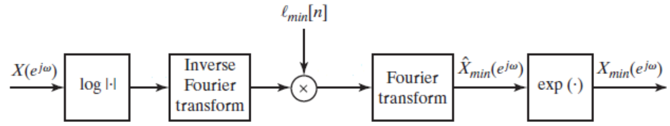

Fig. 4. Derivation of the minimum-phase equivalent system.

the $\mathrm{NTF}_{\mathrm{MSLA}}$ magnitude dataset since it exhibits the same magnitude, with a non-zero phase function, i.e.

$$|X_{ap}(e^{j\omega})| = \frac{|X(e^{j\omega})|}{|X_{min}(e^{j\omega})|} = 1 \tag{8}$$

$$\angle X_{min}(e^{j\omega}) = \mathrm{Im}\left\{\hat{X}_{min}(e^{j\omega})\right\}. \tag{9}$$

The procedure for the derivation of the equivalent dataset is summarized in Fig. 4, where $\ell_{min}[n] \equiv 2u[n] - \delta[n]$. So, the initial dataset that contained magnitude-only measurements, $\left(e^{j\omega_k}, |f_k|\right)$, has been transformed to the equivalent dataset $\left(e^{j\omega_k}, f_k = |f_k|\angle X_{min}(k)\right)$.

2) Vector Fitting Algorithm Details: The Vector Fitting algorithm attempts to fit a rational model to a set of calculated frequency responses $f(s)$, $s = j\omega_k$, $k = 1, 2, \ldots, N_s$, i.e.

$$\sum_{n=1}^{N} \frac{c_n}{s - a_n} + d \approx f(s). \tag{10}$$

The coefficients c_n, a_n are real or complex conjugate pairs, while d is real. The frequency responses $f(s)$ are the ones calculated for the minimum-phase magnitude-equivalent system in the previous subsection. The approximation procedure is formulated as the least squares problem of solving iteratively the pole relocation system

$$\sum_{n=1}^{N} \frac{c_n}{s - a_n^{(i)}} + d = \left[\sum_{n=1}^{N} \frac{\bar{c}_n}{s - a_n^{(i)}} + 1\right] f(s) \tag{11}$$

where i is the iteration index. Considering the parameters c_n, d and $\bar{c}_n$, the system is linear and is solved by evaluating (11) at the N_s frequency data points.

Equation (11) can be solved with iterative numerical methods, replacing the initial poles with the relocated ones at each iteration. The relocated poles are calculated as the eigenvalues of matrix $\mathbf{A}$, i.e. $\left\{a_n^{(i+1)}\right\} = \mathrm{eig}\left\{\mathbf{A} - \mathbf{b}\mathbf{c}^T\right\}$, where $\mathbf{A}$ is an $N \times N$, real-valued, block diagonal matrix holding the poles of the previous iteration, $\mathbf{b}$ is a column vector of ones and $\mathbf{c}^T$ is a row vector holding the residues $\bar{c}_n$. Special attention is needed in the case of complex conjugate poles [5].

Stability is ensured by flipping the poles to the left half-plane, i.e. $a_n^{(i+1)} = -a_n^{(i+1)}$, if they are unstable, i.e. $\mathrm{Re}\left\{a_n^{(i+1)}\right\} > 0$. After convergence, the computed values for the poles are used to compute the residues of the rational model, as well as d, by solving (10) in the least squares sense. This is an overdetermined linear system, solved in the same manner as the pole relocation system. The resulting continuous time model is transformed into a discrete time one via the pole-zero matching technique [7].

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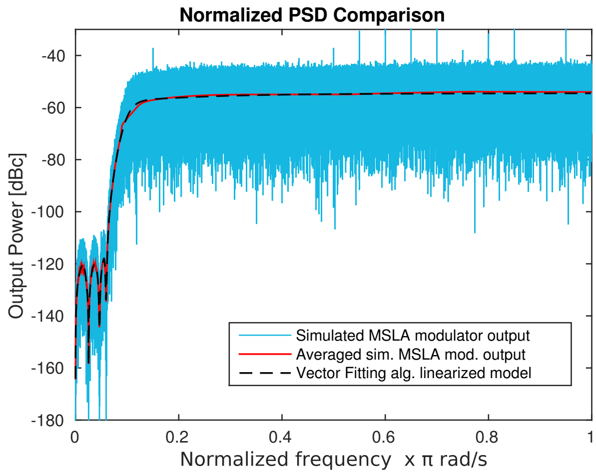

Fig. 5. Output power spectrum of a low-pass MSLA modulator and its linearized model.

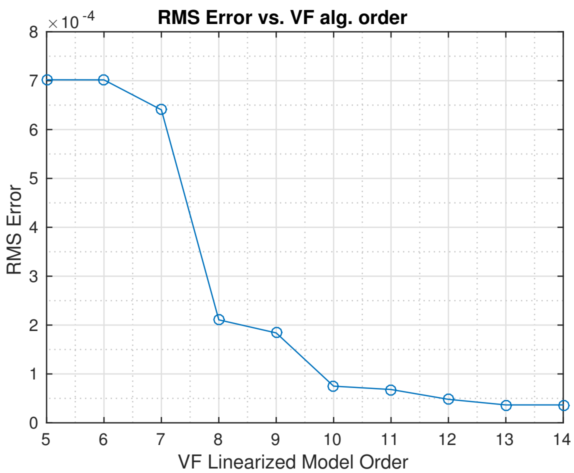

Fig. 6. RMS fit error vs. VF algorithm linearized system order.

B. Simulation Results

The effectiveness of the Vector Fitting (VF) algorithm described previously is showcased in the following simulation results. First, let us assume a low-pass MSLA modulator with $r = k = 10$ and $p = 2$. The input to the modulator is chosen to be a sinusoidal signal of amplitude 0.43. The 7-th order NTF is obtained by the Delta Sigma Toolbox [8] with parameters OSR $= 16$, $||NTF||_\infty = 2$ and use of optimized zeros. It should be noted that the identification procedure requires averaging of the spectrum data to achieve proper results and fast convergence with relatively small data sets. The simulated output spectrum of the modulator along with its averaged version and the VF linearized model are shown in Fig. 5. The linearized model is obtained from the VF algorithm using model order $N = 15$ and 200 iterations. The dataset length is 700 points. There is a nearly perfect match between the linearized model obtained from the VF algorithm and the averaged simulated MSLA modulator output spectrum.

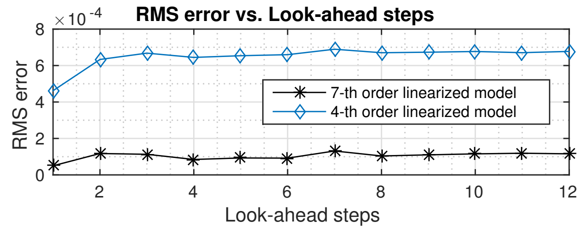

Fig. 7. RMS fit error vs. look-ahead steps k.

As shown in Fig. 6 there is a significant error in the linearized model if the model order is less that 8. This makes sense since the NTF is of order 7. The slight decrease in the RMS error for orders greater than 8 is due to the better fitting achieved for the unfiltered spurs of the averaged simulated spectrum. Analogous results are obtained for band-pass MSLA modulators and for comparison filters of different orders.

In Fig. 7 the influence of the number of look-ahead steps k on the RMS fit error is investigated. The results are obtained for a 3-rd order low-pass MSLA modulator. It is concluded that the number of look-ahead steps k does not have an impact on the accuracy of the linearized model for a specific model order. However, as it was observed in the previous simulation results in Fig. 6, a higher linearized model order results in lower RMS fit error.

IV. Conclusion

A method for the derivation of the MSLA modulator linearized system transfer functions has been presented. The basics of MSLA modulators were introduced and the problem of their analysis using conventional methods was highlighted. It was shown that system identification methods, such as the Vector Fitting method used here, can accurately model the nonlinear MSLA modulator system as a linearized LTI one. Simulation results supporting the proposed methodology were also presented. Further refinement of our methods is due, as well as stability analysis of the MSLA modulator using the linearized models along with techniques used for stability analysis of conventional ΣΔ modulators.

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