Spectral Properties of Dithered Nyquist-rate Single-Bit Quantized Amplitude-Modulated Sinewaves

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***Abstract*—The spectrum of Nyquist-rate, single-bit quantized, randomly-dithered, amplitude-modulated, sinewave is studied analytically. It is shown that by choosing the dithering sequence appropriately and imposing certain constraints on the modulating sequence, the output is spurs-free and the quantization noise is white. MATLAB simulation confirms the theoretical results.**

Keywords—Direct digital synthesis, frequency spurs, amplitude modulation, quantization, spectrum

I. INTRODUCTION

Digital-RF architectures have been developed over the past few years to provide alternatives to the classical analog and mixed-signal ones in RF integrated circuits. They offer the advantages of digital architectures, i.e., design and layout automation, verification capability, easy migration to newer technologies; and, most importantly, faster operation with lower power and lesser area requirements with technology down-scaling [1].

Single-bit digital signaling is instrumental in several digital-RF architectures especially in all-digital frequency synthesis [2]. The work in [2][3]-[6] considered single-bit Nyquist-rate sinewave quantization as an approach to generating single-bit digital waveforms with sinewave-like spectrum and use them as carrier RF signals. The case is similar to that of a Direct Digital Synthesizer (DDS) but with a 1-Bit output, Nyquist-rate Digital to Analog Converter (DAC). Here, however, the phase and/or amplitude dithering is very critical to suppress or, eliminate the (otherwise very strong) quantization frequency spurs.

This work extends the results in [2][3]-[4] towards amplitude-modulated sinewaves. It studies the Power Spectral Density (PSD) of Nyquist-rate dithered single-bit-quantized amplitude-modulated sinewaves when the dithering sequence and certain constrains on the modulating sequence are imposed to guaranty spurs-free output.

The concept studied in the paper is illustrated in Fig. 1. A random sequence $\mathbf{u}_k$ (dither) is subtracted from the Amplitude-Modulated sinewave $A_k \cos(\Omega k)$ and the difference is quantized resulting in output sequence

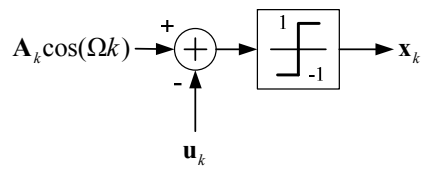

Fig. 1: Dithered single-bit quantized amplitude-modulated sinewave

$$\mathbf{x}_k = \operatorname{sgn}\left(A_k \cos(\Omega k) - \mathbf{u}_k\right) \tag{1}$$

which can be written as

$$\mathbf{x}_k = \begin{cases} 1 & \text{if } \mathbf{u}_k < A_k \cos(\Omega k) \\ -1 & \text{if } \mathbf{u}_k > A_k \cos(\Omega k) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The modulating sequence $\{A_k\}$ may be deterministic or random, indicated with bold symbol $\{\mathbf{A}_k\}$. Both cases are considered in the paper. The carrier frequency (in the discrete-time reference) is Ω with $0 < \Omega < \pi$.

Considering $A_k \cos(\Omega k)$ as the desirable output, we define the output error as

$$\mathbf{e}_k = \mathbf{x}_k - A_k \cos(\Omega k) \tag{3}$$

and our goal is to determine the spectral properties of the error sequence $\{\mathbf{e}_k\}$.

II. ASSUMPTIONS AND PRELIMINARY RESULTS

Throughout the paper, we assume that the dithering sequence $\{\mathbf{u}_k\}$ is formed of independent random variables which are uniformly distributed in $[-1,1]$. Then, the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) [7] of the dither are $f(u) = 1/2$ and $F(u) \triangleq \Pr(\mathbf{u}_k \le u) = (1+u)/2$, respectively for $u \in [-1,1]$. Also, $\Pr(-u < \mathbf{u}_k < u) = u$ for every $u \in [0,1]$. Regarding the amplitude sequence, $\{A_k\}$, deterministic or random, we assume that $A_k \in [-1,1]$ for every $k \in \mathbb{Z}$.

A. Preliminary Results

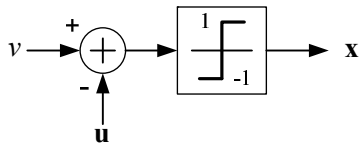

Figure 2: Quantization of a fixed number v with dithering $\mathbf{u}$

To simplify notation, consider the case in Fig. 2, let $v \in [-1,1]$ be a given number and $\mathbf{u}$ be a random variable uniformly distributed in $[-1,1]$. Note that $\mathbf{x} = \operatorname{sgn}(v - \mathbf{u})$ is also a random variable with $\mathbf{x} \in \{\pm 1\}$; the probability of $\mathbf{x} = 0$ is zero and is ignored.

For every $v \in [-1,1]$ it is $\Pr(\mathbf{x} = 1) = \Pr(\mathbf{u} < v)$ as well as $\Pr(\mathbf{x} = -1) = 1 - \Pr(\mathbf{u} < v)$ giving

$$\begin{aligned} \Pr(\mathbf{x} = 1) &= (1+v)/2 \\ \Pr(\mathbf{x} = -1) &= (1-v)/2 \end{aligned} \tag{4}$$

Using Eqs. (4) we derive the expected value of $\mathbf{x}$, $E\{\mathbf{x}\} = \Pr(\mathbf{x} = 1) - \Pr(\mathbf{x} = -1)$, to be $E\{\mathbf{x}\} = v$, i.e., $E\{\mathbf{x}\}$ equals the input value. The output quantization error is $\mathbf{e} = \mathbf{x} - v$ and its expected value is $E\{\mathbf{e}\} = 0$ implying also $E\{\mathbf{e}v\} = 0$. Finally, the variance of the error is $E\{\mathbf{e}^2\} = E\{\mathbf{x}^2 - 2\mathbf{x}v + v^2\} = 1 - v^2$.

Regarding the range of values of the error, $\mathbf{e}$, we note that it takes two possible values $\pm 1 - v$ with probability $(1 \pm v)/2$ respectively. Therefore $\mathbf{e} \in (-2,2)$ for every $v \in [-1,1]$. We summarize the results below.

<u>Lemma 1</u>: Let $v \in [-1,1]$ be a fixed number, $\mathbf{u}$ be uniformly distributed in $[-1,1]$, $\mathbf{x} = \operatorname{sgn}(v - \mathbf{u})$ and $\mathbf{e} = \mathbf{x} - v$. Then $\mathbf{e} \in (-2,2)$ and

$$E\{\mathbf{x}\} = v,\ E\{\mathbf{e}\} = 0,\ E\{\mathbf{e}v\} = 0,\ E\{\mathbf{e}^2\} = 1 - v^2. \tag{5}$$

III. PSD OF SINGLE-BIT QUANTIZED AM SINEWAVE

The Power Spectral Density (PSD) of a Wide-Sense Stationary (WSS) random sequence is the Discrete-Time Fourier Transform (DTFT) of its autocorrelation function [7]

$$r_{\mathbf{x}}(n,m) = E\{\mathbf{x}_n \mathbf{x}_m\}. \tag{6}$$

Here however the output $\{\mathbf{x}_k\}$ and error $\{\mathbf{e}_k\}$ random sequences are not WSS because of their time dependence on the deterministic $A_k \cos(\Omega k)$ or random $\mathbf{A}_k \cos(\Omega k)$ input. In such cases the autocorrelation function can be replaced by the time-average autocorrelation function [8] defined by Eq. (7) below which is valid for a wide class of random sequences.

$$\overline{r}_{\mathbf{x}}(k) \triangleq \lim_{M \to \infty} \frac{1}{2M+1} \sum_{m=-M}^{M} r_{\mathbf{x}}(k+m, m) \tag{7}$$

The PSD of output $\{\mathbf{x}_k\}$ is the DTFT of $\{\overline{r}_{\mathbf{x}}(k)\}$, i.e.,

$$s_{\mathbf{x}}(\omega) = \sum_{k=-\infty}^{\infty} \overline{r}_{\mathbf{x}}(k) e^{-ik\omega} \tag{8}$$

For notational convenience we use $\overline{r_{\mathbf{x}}(k+m,m)}^{m}$ to indicate the time-averaging operation with respect to variable m. When the averaging variable is obvious, it is omitted.

We consider first the case of deterministic input sequence $\{A_k \cos(\Omega k)\}$ and derive the output PSD as a sum of the PSD of the input (desirable signal) and noise PSD. Then we expand the results to the case of random input sequence $\{\mathbf{A}_k \cos(\Omega k)\}$.

A. Deterministic Amplitude Modulation

Let $\{A_k\}$ with $A_k \in [-1,1]$ be a deterministic sequence, $v_k = A_k \cos(\Omega k)$, $\mathbf{x}_k = \operatorname{sgn}(v_k - \mathbf{u}_k)$ and $\mathbf{e}_k = \mathbf{x}_k - v_k$. Since sequence $\{\mathbf{u}_k\}$ is formed of independent random variables so do sequences $\{\mathbf{x}_k\}$ and $\{\mathbf{e}_k\}$. Therefore $E\{\mathbf{e}_n \mathbf{e}_m\} = E\{\mathbf{e}_n\} E\{\mathbf{e}_m\}$ and since $E\{\mathbf{e}_k\} = 0$ from Lemma 1, we get that $E\{\mathbf{e}_n \mathbf{e}_m\} = 0$ when $n \neq m$. Again, Lemma 1 implies $E\{\mathbf{e}_k^2\} = 1 - A_k^2 \cos^2(\Omega k)$. Therefore,

$$\begin{aligned} \overline{r}_{\mathbf{e}}(k) &= \lim_{M \to \infty} \frac{1}{2M+1} \sum_{m=-M}^{M} E\{\mathbf{e}_{k+m} \mathbf{e}_m\} \\ &= \left[\lim_{M \to \infty} \frac{1}{2M+1} \sum_{m=-M}^{M} E\{\mathbf{e}_m^2\} \right] \cdot \delta_k \\ &= \left[1 - \overline{A_k^2 \cos^2(\Omega k)} \right] \cdot \delta_k \\ &= \left[1 - \frac{1}{2} \overline{A_k^2} - \frac{1}{2} \overline{A_k^2 \cos(2\Omega k)} \right] \cdot \delta_k \end{aligned} \tag{9}$$

To simplify Eq. (9) we can assume that the DTFT, $A(\omega) = \sum_{k=-\infty}^{\infty} A_k e^{-ik\omega}$, of $\{A_k\}$ exists and is band-limited by Ω_{A} with $0 < \Omega_{\mathrm{A}} < \pi/2$, i.e, $A(\omega) = 0$ for $\Omega_{\mathrm{A}} < \omega < \pi$. Then by the properties of DTFT[9], $A_2(\omega) \triangleq \sum_{k=-\infty}^{\infty} A_k^2 e^{-ik\omega}$, i.e. the DTFT of $\{A_k^2\}$, is band-limited by $2\Omega_{\mathrm{A}}$, [9]. If it is

$$\Omega_{\mathrm{A}} < \Omega < \pi - \Omega_{\mathrm{A}} \tag{10}$$

then $A_2(2\Omega)=0$ and moreover $\overline{A_k^2\cos(2\Omega k)}=0$ simplifying (9) to $\overline{r}_{\mathbf{e}}(k)=(1-P_A/2)\cdot\delta_k$ where $P_A=\overline{A_k^2}$ is the average power of A_k. Therefore the PSD of the quantization error is the flat noise floor

$$s_{\mathbf{e}}(\omega)=1-\frac{P_A}{2}. \tag{11}$$

Now, since $\mathbf{x}_k=\mathbf{e}_k+v_k$, and $E\{\mathbf{e}_n v_m\}=0$ for every n,m from Lemma 1 and the independence of $\{\mathbf{u}_k\}$, we get that $\overline{r}_{\mathbf{x}}(k)=\overline{r}_{\mathbf{v}}(k)+\overline{r}_{\mathbf{e}}(k)$ and so

$$s_{\mathbf{x}}(\omega)=s_{\mathbf{v}}(\omega)+s_{\mathbf{e}}(\omega). \tag{12}$$

B. Random Amplitude Modulation

Now we consider the case of random amplitude modulating sequence $\{\mathbf{A}_k\}$ and assume it is WSS with expected value $E\{\mathbf{A}_k\}=A$ and autocorrelation function $r_{\mathbf{A}}(n-m)\triangleq E\{\mathbf{A}_n\mathbf{A}_m\}$. Following our definitions it is

$$\mathbf{v}_k=\mathbf{A}_k\cos(\Omega k), \tag{13}$$

$$\mathbf{x}_k=\operatorname{sgn}(\mathbf{v}_k-\mathbf{u}_k) \tag{14}$$

and again the error is defined by

$$\mathbf{e}_k=\mathbf{x}_k-\mathbf{v}_k. \tag{15}$$

Note that $\mathbf{x}_k$ and $\mathbf{e}_k$ are functions of both random variables $\mathbf{A}_k$ and $\mathbf{u}_k$, and are independent of $\mathbf{u}_m$, for $m\neq k$.

In the following analysis we use the identity of the conditional expected value [7].

$$E\{E\{\mathbf{Y}|\mathbf{X}\}\}=E\{\mathbf{Y}\} \tag{16}$$

First we derive $E\{\mathbf{v}_n\mathbf{e}_m|\mathbf{A}_n,\mathbf{A}_m\}$ observing that here $\mathbf{A}_n,\mathbf{A}_m$ and so $\mathbf{v}_n,\mathbf{v}_m$ are treated as real numbers, so

$$E\{\mathbf{v}_n\mathbf{e}_m|\mathbf{A}_n,\mathbf{A}_m\}=\mathbf{v}_nE\{\mathbf{e}_m|\mathbf{A}_n,\mathbf{A}_m\}=\mathbf{v}_nE\{\mathbf{e}_m|\mathbf{A}_m\}=0$$

where in the second step we used Eq. (5) of Lemma 1. And so from Eq. (16), $E\{\mathbf{v}_n\mathbf{e}_m\}=E\{E\{\mathbf{v}_n\mathbf{e}_m|\mathbf{A}_n,\mathbf{A}_m\}\}=0$ for every m,n. Then

$$\begin{aligned}r_{\mathbf{x}}(n,m)&=E\{\mathbf{x}_n\mathbf{x}_m\}\\&=E\{(\mathbf{v}_n+\mathbf{e}_n)(\mathbf{v}_m+\mathbf{e}_m)\}\\&=E\{\mathbf{v}_n\mathbf{v}_m\}+E\{\mathbf{e}_n\mathbf{e}_m\}+E\{\mathbf{v}_n\mathbf{e}_m\}+E\{\mathbf{v}_m\mathbf{e}_n\}\end{aligned}$$

where the last two summands are zero implying that $r_{\mathbf{x}}(n,m)=r_{\mathbf{v}}(n,m)+r_{\mathbf{e}}(n,m)$ and so

$$\overline{r}_{\mathbf{x}}(k)=\overline{r}_{\mathbf{v}}(k)+\overline{r}_{\mathbf{e}}(k) \tag{17}$$

which via DTFT results in

$$s_{\mathbf{x}}(\omega)=s_{\mathbf{v}}(\omega)+s_{\mathbf{e}}(\omega). \tag{18}$$

To derive $s_{\mathbf{x}}(\omega)$ we note first that for every m,n with $m\neq n$, $\mathbf{x}_n,\mathbf{x}_m$ are conditionally independent given $\mathbf{A}_n,\mathbf{A}_m$ (and therefore with respect to $\mathbf{v}_n,\mathbf{v}_m$). This implies that $E\{\mathbf{x}_n\mathbf{x}_m|\mathbf{A}_n,\mathbf{A}_m\}=E\{\mathbf{x}_n|\mathbf{A}_n\}E\{\mathbf{x}_m|\mathbf{A}_m\}$ and from Lemma 1 we have $E\{\mathbf{x}_k|\mathbf{A}_k\}=\mathbf{v}_k$. So $E\{\mathbf{x}_n\mathbf{x}_m|\mathbf{A}_n,\mathbf{A}_m\}=\mathbf{v}_n\mathbf{v}_m$ implying

$$\begin{aligned}E\{\mathbf{x}_n\mathbf{x}_m\}&=E\{E\{\mathbf{x}_n\mathbf{x}_m|\mathbf{A}_n,\mathbf{A}_m\}\}\\&=E\{\mathbf{v}_n\mathbf{v}_m\}\\&=E\{\mathbf{A}_n\cos(\Omega n)\mathbf{A}_m\cos(\Omega m)\}\\&=E\{\mathbf{A}_n\mathbf{A}_m\}\cos(\Omega n)\cos(\Omega m)\end{aligned}.$$

Therefore $E\{\mathbf{x}_n\mathbf{x}_m\}=r_{\mathbf{A}}(n-m)\cos(\Omega n)\cos(\Omega m)$ which if we set $n=m+k$, with $k\neq 0$, gives

$$E\{\mathbf{x}_{m+k}\mathbf{x}_m\}=r_{\mathbf{A}}(k)\left[\frac{\cos(\Omega k)}{2}+\frac{\cos(\Omega(2m+k))}{2}\right]$$

Since $\overline{r}_{\mathbf{x}}(k)\triangleq\overline{E\{\mathbf{x}_{m+k}\mathbf{x}_m\}}^m$ and the second cosine averages to zero with respect to m we conclude that

$$\overline{r}_{\mathbf{x}}(k)=\frac{1}{2}r_{\mathbf{A}}(k)\cos(\Omega k) \tag{19}$$

for $k\neq 0$. Now, since $\mathbf{x}_k\in\{-1,1\}$ with probability one, $E\{\mathbf{x}_n^2\}=1$. Therefore,

$$\overline{r}_{\mathbf{x}}(k)=\frac{1}{2}r_{\mathbf{A}}(k)\cos(\Omega k)+\left[1-\frac{1}{2}r_{\mathbf{A}}(0)\right]\delta_\kappa \tag{20}$$

Now, from Eq. (13) we get directly that for every m,n, $E\{\mathbf{v}_n\mathbf{v}_m\}=E\{\mathbf{A}_n\mathbf{A}_m\}\cos(\Omega n)\cos(\Omega m)$ implying similarly to Eq. (19) that

$$\overline{r}_{\mathbf{v}}(k)=\frac{1}{2}r_{\mathbf{A}}(k)\cos(\Omega k). \tag{21}$$

The spectrum of $\{\mathbf{v}_k\}$ (i.e. the desirable part of the output signal) is derived via the DTFT of Eq. (21),

$$s_{\mathbf{v}}(\omega)=\frac{1}{2}\left[s_{\mathbf{A}}(\omega-\Omega)+s_{\mathbf{A}}(\omega+\Omega)\right] \tag{22}$$

Finally, using Eq. (17) we get that

$$\overline{r}_{\mathbf{e}}(k)=\left[1-\frac{1}{2}r_{\mathbf{A}}(0)\right]\delta_\kappa. \tag{23}$$

and taking the DTFT of it gives

$$s_{\mathbf{e}}(\omega) = 1 - \frac{1}{2} r_{\mathbf{A}}(0) \tag{24}$$

We observe that $r_{\mathbf{A}}(0) \triangleq E\{\mathbf{A}_n^2\}$ is the expected power of the modulating sequence $\{\mathbf{A}_k\}$ corresponding exactly to P_A of the deterministic case captured by Eq. (11).

IV. Examples

The following examples, with deterministic and random modulation, illustrate the developed theory and introduce the definition of dynamic range.

A. Sinusoidal Amplitude Modulation of the Carrier

We consider first the deterministic case of sinusoidal modulation $A_k = \alpha + \beta \sin(\Omega_{\text{mod}} k)$ with parameters α, β such that $\alpha, \beta > 0$ and $\alpha + \beta \le 1$. Then $P_A = \overline{A_k^2} = \alpha^2 + \frac{\beta^2}{2}$ which combined with Eq. (11) imply $s_{\mathbf{e}}(\omega) = 1 - \frac{\alpha^2}{2} - \frac{\beta^2}{4}$. Also, from $v_k = A_k \cos(\Omega k)$ we get that

$$v_k = \alpha \cos(\Omega k) + \frac{\beta}{2}\left[\sin\left((\Omega + \Omega_{\text{mod}})k\right) - \sin\left((\Omega - \Omega_{\text{mod}})k\right)\right]$$

To have a power reference signal we assume for simplicity that $\alpha \ge \beta/2$ and so the power of the carrier signal $\alpha \cos(\Omega k)$ is higher than that of the sidebands.

Now we define the discrete-time Dynamic Range of the output signal as the ratio of the power of our reference signal here, $\alpha \cos(\Omega k)$, over the PSD of the noise floor. It is $DR_{DT} = \frac{(\alpha^2/4)}{s_{\mathbf{e}}} = \frac{\alpha^2}{4 - 2\alpha^2 - \beta^2}$ where the factor 1/4 of α^2 is because we have to calculate the signal power within only the $[0, \pi]$ part of the spectrum. To convert it to the continuous-time domain, we assume that the sampling frequency is f_s and the resolution bandwidth (of the spectrum analyzer we observe the signal) is *RBW*, [4]. Then the observed dynamic range in logarithmic scale is

$$DR_O = 10 \log_{10}\left(DR_{DT} \cdot \frac{f_s}{RBW}\right) \quad \text{(dB)} \tag{25}$$

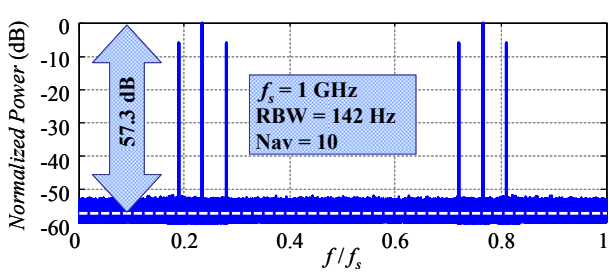

Fig. 3: $s_{\mathbf{x}}(\omega)$ when $A_k = 1/2 + (1/2)\sin(\Omega_{\text{mod}} k)$

For $\Omega = 2\pi \cdot (15/64)$, $\Omega_{\text{mod}} = 2\pi \cdot 0.045$, $\alpha = \beta = 1/2$, $f_s = 1GHz$ and *RBW* = 142Hz, from Eq. (25) we get $DR_O = 57.3$ dB. The output spectrum is shown in Fig. 3.

For $\Omega = 2\pi \cdot 0.135$, $\Omega_{\text{mod}} = 2\pi \cdot 0.03$, $\alpha = 3/4$, $\beta = 1/4$, $f_s = 1GHz$ and *RBW* = 908.3Hz, from Eq. (25) we get $DR_O = 53.4$ dB. The output spectrum is shown in Fig. 4.

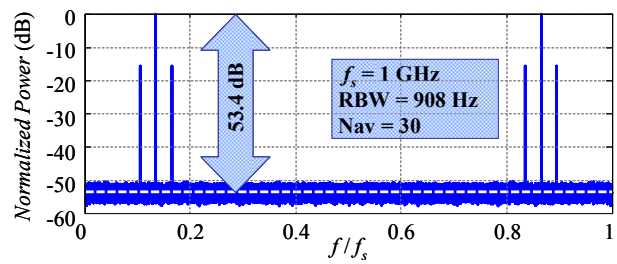

Fig. 4: $s_{\mathbf{x}}(\omega)$ when $A_k = 3/4 + (1/4)\sin(\Omega_{\text{mod}} k)$

B. Random Amplitude Modulation of the Carrier

Here we consider the spectrum of $\mathbf{v}_k = \mathbf{A}_k \cos(\Omega k)$, $\mathbf{x}_k = \text{sgn}(\mathbf{v}_k - \mathbf{u}_k)$ and $\mathbf{e}_k = \mathbf{x}_k - \mathbf{v}_k$ when the modulating random sequence $\{\mathbf{A}_k\}$ is such that $\mathbf{A}_k = \mathbf{B}_{k \bmod 100}$ for all $k \in \mathbb{Z}$ and $\{\mathbf{B}_n\}$ is formed of independent random variables with $\Pr(\mathbf{B}_n = 1) = \Pr(\mathbf{B}_n = -1) = 1/2$. Also $\Omega = 2\pi \cdot (0.26)$ $f_s = 1GHz$ and *RBW* = 10 kHz.

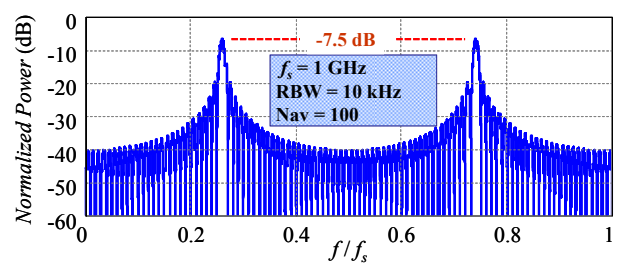

Fig. 5: The spectrum $s_{\mathbf{v}}(\omega)$ of the desirable signal $\mathbf{v}_k = \mathbf{A}_k \cos(\Omega k)$

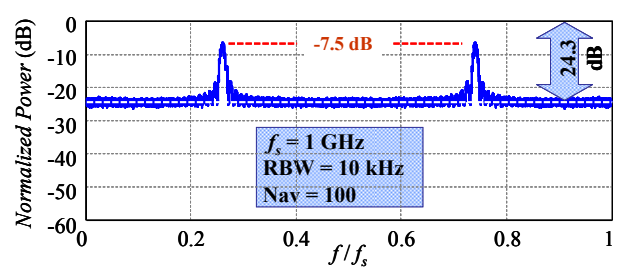

Fig. 6: The spectrum $s_{\mathbf{x}}(\omega)$ of the quantized output signal $\mathbf{x}_k = \text{sgn}(\mathbf{v}_k - \mathbf{u}_k)$. Note that $s_{\mathbf{x}}(\omega) = s_{\mathbf{v}}(\omega) + s_{\mathbf{e}}(\omega)$.

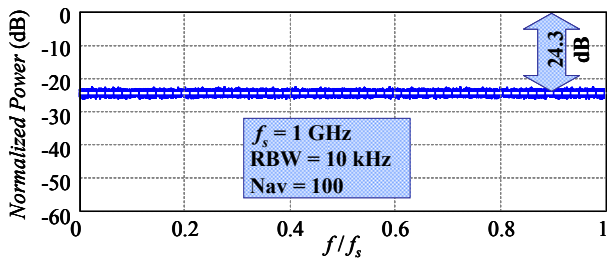

Fig. 7: The spectrum $s_{\mathbf{e}}(\omega)$ of the quantization error $\mathbf{e}_k = \mathbf{x}_k - \mathbf{v}_k$

Figures 5, 6 and 7 show the PSD of $\{\mathbf{v}_k\}$, $\{\mathbf{x}_k\}$ and $\{\mathbf{e}_k\}$ respectively. Observe that $s_{\mathbf{x}}(\omega) = s_{\mathbf{v}}(\omega) + s_{\mathbf{e}}(\omega)$ as stated by Eq. (18).

V. CONCLUSIONS

This work has studied analytically the spectrum of Nyquist-rate, single-bit quantized, randomly-dithered, amplitude-modulated, sinewave. Both cases of deterministic and random amplitude modulation were considered. The spectrum of the quantized output was derived and, under certain assumptions, it is equal to the sum of the input spectrum plus white noise due to quantization error. The choice of dithering and certain constraints on the amplitude and modulation result in spurs-free quantized output.

ACKNOWLEDGEMENT

Described work was partially supported by Broadcom Foundation USA

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