All-Digital Frequency Synthesis based on Single-Bit Nyquist-Rate Sinewave Quantization with IID Random Dithering

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***Abstract*—All-digital frequency synthesis based on single-bit, Nyquist-rate, quantization of sinewave with independent and identically distributed random amplitude dithering is proposed. The output spectrum of the quantizer is derived in closed form and is related to the distribution of the random dither. Conditions for spurs-free output are derived, and the output dynamic range is defined. MATLAB simulation examples illustrate the results of the proposed approach.**

Keywords—Digital-to-frequency converter, direct digital synthesis, frequency spurs, quantization

I. Introduction

The interest in all-digital frequency synthesis (FS) has been intensified in the R.F.I.C. industry over the past few years due to the increasing challenge in the design and the extra cost of fabrication of R.F. analog and mixed-signal I.C. versus standard digital ones in modern nano-scale I.C. technologies, e.g. [1]-[3]. Efforts towards all-digital FS can be traced at least thirty years back [4][6].

Replacing a complex analog or mixed-signal frequency synthesizer with a fully digital one can result in faster concept to market cycle, lower design effort and cost and the advantage of using digital design and verification tools.

Moreover, synchronous single-bit digital outputs of all-digital synthesizers, having sinewave-like spectrum can be used as local-oscillator signals in RF chains and, they can be amplified for transmission or internal use without distortion and with very high efficiency using a switching amplifier. Also one can incorporate digital or analog phase, frequency and amplitude modulation directly in an all-digital synthesizer [7].

The proposed single-bit-output Nyquist-rate sinewave quantization scheme is practically realized by a Direct Digital Synthesizer (DDS) with a 1-Bit output Nyquist-rate Digital to Analog Converter (DAC). Dither is added to the output of the Look-Up-Table (LUT) before the hard quantization to alleviate the nonlinearity and suppress the output spurs by breaking the periodicity of the truncation error.

Note that the DAC is only virtually there representing the act of signum function, or similarly that of an MSB extractor, applied to the sum of the LUT's output with the dither.

In the above setup, single-bit quantization alone, i.e. without dithering, typically results in dense and high-power frequency spurs making the output single-bit digital signal unusable for analog and R.F. applications (although it can by used for clocking digital circuitry).

The paper derives analytically the spectrum of the single-bit dithered quantizer as a function of the dither's Cumulative Distribution Function (CDF) when the dithering sequence is formed of Independent and Identically Distributed (IID) random variables. Moreover, the noise floor power due to random dithering is derived analytically and the output dynamic range is defined and calculated explicitly.

II. Definitions and Assumptions

Random dithering is commonly used to suppress the spurs and shape the noise of quantization in DDS [8] and data converters [9] and to eliminate periodic patterns in fractional-N frequency dividers [10].

Here we consider the extreme case of amplitude dithered DDS with single-bit output quantization (DAC) without oversampling [9] shown in Figure 1. The cosine can be generated using a phase accumulator and a LUT, and the Zero Order Hold (ZOH) outputs a continuous-time single-bit digital waveform. All blocks are clocked by a clock reference of frequency $f_S = 1/T_S$.

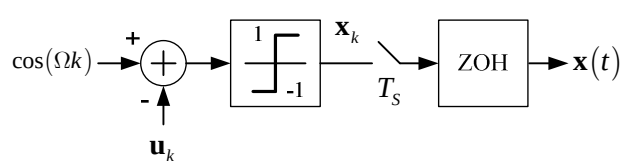

Figure 1: Dithered single-bit quantization of a sinewave

The dithering random sequence $\{\mathbf{u}_k\}$ is subtracted from the sinewave resulting in the discrete-time single-bit (± 1) signal $\mathbf{x}_k = \mathrm{sgn}\left(\cos(\Omega k) - \mathbf{u}_k\right)$ where "sgn" is the signum function.

Throughout the paper we assume that the random sequence $\{\mathbf{u}_k\}$ is formed of IID random variables having CDF $G:[-1,1]\to[0,1]$ which is continuous and has continuous second derivative in $[-1,1]$. Therefore, for every $k\in\mathbb{Z}$ and $u\in[-1,1]$ it is $\Pr(\mathbf{u}_k\le u)=G(u)$, so implicitly we assume that essentially $\mathbf{u}_k$ takes values only within $[-1,1]$. This is plausible because $\cos(\Omega k)$ does the same and hence any larger value range of the dither would be unnecessary.

It also makes sense from an application perspective to assume that $\Omega/(2\pi)$ is rational, i.e. $\Omega=2\pi w/q$ for some integer w such that $0<w<q/2$. In this case

$$\mathbf{x}_k=\operatorname{sgn}\left(\cos\left(2\pi wk/q\right)-\mathbf{u}_k\right) \tag{1}$$

and since $\{\mathbf{u}_k\}$ is an IID random sequence and $\cos(2\pi wk/q)$ has period $q/\gcd(q,w)$, the random sequence $\{\mathbf{x}_k\}$ is cyclostationary of the same period. Since q is a multiple of $q/\gcd(q,w)$, $\{\mathbf{x}_k\}$ is also cyclostationary of period q and we consider it as such in the rest of the paper to simplify notation.

A. *The Period-Average Autocorrelation of* $\{\mathbf{x}_k\}$

The Power Spectral Density (PSD) of a discrete-time wide-sense stationary (WSS) process is the Discrete-Time Fourier Transform (DTFT) of its autocorrelation function [11]

$$r_x(n,m)=E\{\mathbf{x}_n\mathbf{x}_m\}. \tag{2}$$

Since $\{\mathbf{x}_k\}$ is not WSS but cyclostationary of period q its PSD, $s_x(\omega)$, is commonly defined as the DTFT of its period-average autocorrelation [11]-[12], i.e., of

$$\bar{r}_x(k)=\frac{1}{q}\sum_{m=0}^{q-1}r_x(k+m,m) \tag{3}$$

and

$$s_x(\omega)=\sum_{k=-\infty}^{\infty}\bar{r}_x(k)e^{-ik\omega}. \tag{4}$$

To calculate $\bar{r}_X(k)$ we express the CDF $G:[-1,1]\to[0,1]$ as a series of Chebyshev polynomials of the first kind, i.e.,

$$G(u)=\frac{1}{2}+\frac{1}{2}\sum_{j=0}^{\infty}a_jT_j(u). \tag{5}$$

The summand and multiplying factor $1/2$ is used to simplify the algebra. Coefficients a_j are derived according to [13]

$$a_0=\frac{2}{\pi}\int_{-1}^{1}\frac{G(u)}{\sqrt{1-u^2}}du-1,\quad a_{j>0}=\frac{4}{\pi}\int_{-1}^{1}\frac{G(u)T_j(u)}{\sqrt{1-u^2}}du. \tag{6}$$

Since G was assumed continuous, its series expansion (5) converges to G everywhere in $[-1,1]$. Inversely, G can be defined using coefficients a_j as long as series (5) converges to a continuous function and G is indeed a CDF. Assuming further that series (5) is term-by-term differentiable, a necessary and sufficient set of conditions for G to be a CDF is

$$G(-1)=0,\quad G(1)=1\quad\&\quad G'(u)\ge 0\quad\forall u\in[-1,1] \tag{7}$$

Since $T_j(\pm1)=(\pm1)^j$, $j=0,1,2,\ldots$, and $T_j'(u)=j\cdot U_{j-1}(u)$, for $j=1,2,3,\ldots$, where U_j is the j-th Chebyshev polynomial of the 2nd kind [13], Eqs. (7) can be written as

$$\begin{aligned}&\sum_{j=0}^{\infty}(-1)^j a_j=-1,\quad \sum_{j=0}^{\infty}a_j=1\\&\sum_{j=1}^{\infty}ja_jU_{j-1}(u)\ge 0\quad\forall u\in[-1,1]\end{aligned}. \tag{8}$$

Section IV illustrates how coefficients a_j are calculated for two cases of CDF G and used to derive the PSD.

III. Power Spectral Density and Dynamic Range

Using the above definitions we can express $\bar{r}_X(k)$ as a function of the coefficients a_j, $j=0,1,2,\ldots$. Specifically, we have that [14] the period-average autocorrelation $\bar{r}_X(k)$ of $\{\mathbf{x}_k\}$ is given by (9) where $\delta_0=1$ and $\delta_{k\ne0}=0$.

$$\bar{r}_x(k)=a_0^2+\sum_{j=1}^{\infty}\frac{a_j^2}{2}\cos\left(\frac{2\pi kjw}{q}\right)+\left(1-a_0^2-\sum_{j=1}^{\infty}\frac{a_j^2}{2}\right)\delta_k \tag{9}$$

Note that $\bar{r}_x$ comprises of DC term a_0^2, harmonics of $\cos(2\pi kw/q)$ and an impulse term at $k=0$. Moreover, the amplitude of the jth harmonic, i.e., $a_j^2/2$, is half the square of the projection of CDF G to the jth Chebyshev polynomial according to Eqs. (6). This implies that by selecting CDF G we can "shape" the period-average autocorrelation function and so the PSD of $\{\mathbf{x}_k\}$. Note however that time is discrete and the harmonics of $\cos(2\pi kw/q)$ in Eq. (9) are subject to aliasing and folding into the frequency domain $\omega\in[0,2\pi)$ simply because $\cos(2\pi kjw/q)=\cos\left(2\pi k\left((jw)\bmod q\right)/q\right)$.

A. *Power Spectral Density of the Output Signal* $\mathbf{x}(t)$

The output $\mathbf{x}(t)$ of the ZOH in Figure 1 is a continuous-time signal which can be written in the form

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty}\mathbf{x}_k p\left(\frac{t}{T_S}-k\right) \tag{10}$$

where $T_S=1/f_S$ is the sampling period and pulse $p(t)$ is 1 for $t\in[0,1)$ and zero otherwise corresponding to ZOH's operation. It can be shown [14] that the PSD of $\mathbf{x}(t)$ is

$$S_x(f)=T_S\cdot\operatorname{sinc}^2(f\cdot T_S)\cdot s_x(2\pi fT_S) \tag{11}$$

where $s_x(\omega)$ is the DTFT in Eq. (4) of the period-average autocorrelation $\overline{r_X}$ and $T_S \cdot \text{sinc}^2(f \cdot T_S)$ is due to the shape of the pulse $p(t)$. Combining Eqs. (4), (9) and (11), and after a lengthy algebraic manipulation we can derive the following Theorem whose proof can be found in [14].

Theorem: Following the above definitions and assumptions and further assuming that $\gcd(w,q)=1$ the PSD of $\mathbf{x}(t)$ is

$$S_x(f) = \text{sinc}^2\left(\frac{f}{f_S}\right) \cdot \left(S_{HA}(f) + S_N(f) + S_{DC}(f) \right) \quad (12)$$

where HArmonics, Noise and DC components are given by

$$S_{HA}(f) = \frac{1}{4}\sum_{h=1}^{\infty} b_h \left(\delta\left(f - \frac{h}{q} f_S \right) + \delta\left(f + \frac{h}{q} f_S \right) \right), \quad (13)$$

$$S_N(f) = \frac{1}{f_S} \cdot \left(1 - a_0^2 - \frac{1}{2}\sum_{j=1}^{\infty} a_j^2 \right), \quad (14)$$

and

$$S_{DC}(f) = \frac{b_0 + 3a_0^2}{4}\delta(f) \quad (15)$$

respectively. For $h = 0,1,2,...$ the power of the frequency components at $\pm(h/q) f_S$ in Eq. (13) is $b_h/4$ and

$$b_h \triangleq \sum_{r=-\infty}^{\infty} a_{I(h,r)}^2 \quad (16)$$

where $I(h,r) = |j_1 h + qr|$; constant j_1 (and k_1 which is not involved in the expression) is derived solving the Diophantine equation $wj_1 + qk_1 = 1$. Specifically coefficient b_w of the frequency components at $\pm(w/q) f_S$ is $b_w = \sum_{r=-\infty}^{\infty} a_{|1+qr|}^2$. □

Observing Eq. (16) we note that the contributions of coefficients a_j to the total power of frequency component at $\pm(h/q) f_S$ are cumulative since $a_{I(h,r)}^2 \geq 0$. Therefore, to minimize the spurs in the output we should zero as many of the coefficients a_j as possible because the smaller the set of nonzero coefficients a_j is, the smaller the set of frequency components present in $S_{HA}(f)$ will be.

Note that in order to derive coefficient b_h we first find a solution (j_1, k_1) of the Diophantine equation $wj_1 + qk_1 = 1$ using the Euclidean algorithm ("gcd" function in MATLAB). This is always possible due to our assumption that $\gcd(w,q)=1$. Any (j_1, k_1) of the infinitely many solutions is acceptable but one with absolutely small j_1, k_1 is convenient. Then we apply Eq. (16) where only j_1 is used in $|j_1 h + qr|$. The procedure is illustrated in the examples of Section IV.

B. *Noise Floor and Dynamic Range*

Typically, the desirable frequency component at the output is at frequency $(w/q) f_S$ with amplitude $\text{sinc}^2(f/f_S) \cdot b_w/4$, captured by $S_{HA}(f)$ in Eq. (13). Also, the output noise (which is the only component of continuous spectrum) has PSD $\text{sinc}^2(f/f_S) \cdot S_N(f)$. We define the Dynamic Range (DR) of the output as the ratio of the signal power to noise's PSD

$$DR = 10\log_{10}\left(\frac{\text{sinc}^2(f/f_S) \cdot b_w/4}{\text{sinc}^2(f/f_S) \cdot S_N(f)} \right) \quad \text{(dB)}$$

and after replacing the values of b_w and $S_N(f)$ we get

$$DR = 10\log_{10}\left(\frac{\sum_{r=-\infty}^{\infty} a_{|1+qr|}^2}{1 - a_0^2 - \sum_{j=1}^{\infty} \frac{a_j^2}{2}} \right) + 10\log_{10}(f_S) - 6.02 \quad \text{(dB)} \quad (17)$$

The definition of the DR can be used for other frequency components of interest as well. Also, note that the summand $10\log_{10}(f_S)$ in Eq. (17) is expected since the power of the sinewave's quantization error is spread over frequency bandwidth proportional to the sampling frequency. The use of Eq. (17) is illustrated in the examples of Section IV.

IV. Examples and Simulations

First we emphasize the importance of dithering in spurs suppression in the case of single-bit quantization by considering the case of $w = 25$ and $q = 64$ without dither. This is the limiting case of the above setup when CDF $G(u)$ is zero for $u \in [-1, 0)$ and one for $u \in (0,1]$, i.e. $\mathbf{u}_k = 0$ with probability one. Figure 2 shows the output spectrum ignoring the weighting factor $\text{sinc}^2(f/f_S)$. The spectrum is the result of MATLAB simulation and coincides completely with the theory, Eq. (12). Note that although G is discontinuous at $u = 0$ its series expansion, Eq. (5), is valid for $u \neq 0$ implying $a_{2k} = 0$ and $a_{2k+1} = 4(-1)^k / ((2k+1)\pi)$ for $k = 0,1,2,...$.

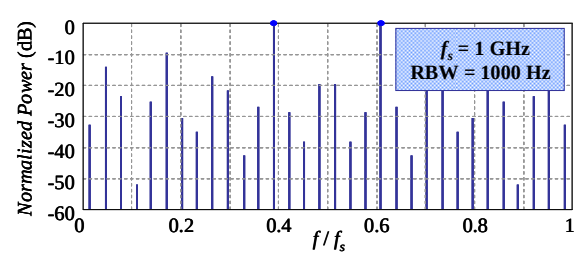

Figure 2: Spectrum of single-bit quantized sinewave without dithering when $w = 25$ and $q = 64$; ignoring the weighting factor $\text{sinc}^2(f/f_S)$.

Now we consider the same case of $w = 25$ and $q = 64$ when the dithering sequence $\{\mathbf{u}_k\}$ is formed of *uniformly*

distributed IID random variables, i.e. the probability density function is constant, $G'(u)=1/2$ in $[-1,1]$ and so the CDF is $G(u)=(u+1)/2$. Since $T_1(u)=u$ we derive by inspection that $a_0=0$, $a_1=1$ and $a_k=0$ for $k=2,3,4,\ldots$.

To derive the coefficients b_h in the Theorem we find a solution of the Diophantine equation $25j_1+64k_1=1$, e.g. $(j_1,k_1)=(-23,9)$. For b_h, $h=0,1,2,\ldots$ to be nonzero, there must exist some $r\in\mathbb{Z}$ for which

$$I(h,r)=|j_1h+64r|=1 \tag{18}$$

This is because $a_1=1$ and $a_0=0$, $a_k=0$ for all $k=2,3,4,\ldots$, therefore only a_1 can contribute to b_h. Since $25j_1+64k_1=1$, a particular solution of $j_1h+64r=1$ is $(h,r)=(25,k_1)$ and so the general solution of Eq. (18) is

$$(h,r)=\pm(25,9)+\rho(64,23),\ \rho\in\mathbb{Z} \tag{19}$$

Since it is $h\geq 0$ we conclude that the (only) non zero coefficients b_h, $h=0,1,2,\ldots$ are b_{25} and $b_{\eta\cdot 64\pm 25}$, $\eta=1,2,3,\ldots$ From Eq. (16) we also get that $b_{25}=b_{\eta\cdot 64\pm 25}=1$, $\eta=1,2,3,\ldots$. So we derive that

$$S_{HA}(f)=\frac{1}{4}\sum_{\substack{h=25,\ \eta\cdot 64\pm 25\\ \eta=1,2,3,\ldots}}\left(\delta\left(f-\frac{h}{q}f_S\right)+\delta\left(f+\frac{h}{q}f_S\right)\right),$$

$S_N(f)=1/(2f_S)$ and $S_{DC}(f)=0$. The result coincides completely with the PSD derived using simulation, shown in Figure 3 where the weighted factor $\mathrm{sinc}^2(fT_S)$ is ignored.

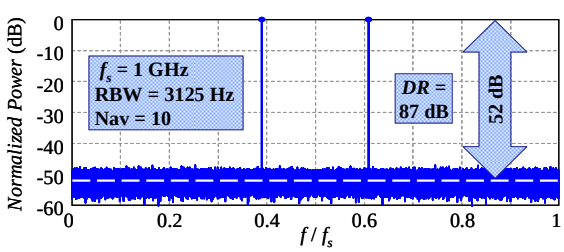

Figure 3: Spectrum of single-bit quantized sinewave with uniformly distributed dither when $w=25$ and $q=64$; the weighting factor $\mathrm{sinc}^2(f/f_S)$ is ignored. $f_S=1\,GHz$, Resolution BW = 3125 Hz and waveform averaging Nav=10 runs.

We note that there are only two frequency components in the frequency range $[0,f_S]$, the desirable one at $(25/64)f_S$ and the image of it at $((64-25)/64)f_S$ due to the discrete time nature of $\{\mathbf{x}_k\}$. The second one cannot be eliminated unless a continuous-time filter is used. Since no spurs are present we conclude that the uniform CDF results in spurs-free output. This is true in general for every integers w and q satisfying our assumptions.

Since $a_0=0$, $a_1=1$ and $a_k=0$ for $k=2,3,4,\ldots$, the dynamic range derived from Eq. (17) is expressed as

$$DR=10\log_{10}(f_S)-\ 3.01\ \text{dB} \tag{20}$$

In the PSD graph of the case $w=25$ and $q=64$ shown in Figure 3 the dashed white line indicates the averaged noise floor power. For $f_S=1\,GHz$ we get $DR\cong 87$ dB. Subtracting $10\log_{10}(RBW)$ dB, where $RBW=3125\,Hz$, to account for the resolution BW used for the simulation we get a very good match to the simulated 52 dB (Figure 3).

V. Conclusions

All-digital frequency synthesis using single-bit, Nyquist-rate quantization of sinewave and random dithering formed of independent and identically distributed random variables has been studied mathematically. The output spectrum has been calculated analytically as a function of the dither's Cumulative Distribution Function. It has been shown that uniformly distributed dither with range equal to that of the sinewave results in spurious-free output spectrum. The noise floor level due to dithering has been calculated analytically and the output dynamic range has been defined and calculated explicitly for the case of uniformly distributed dither. Examples based on MATLAB have been presented to illustrate the theory. The simulation results are in complete agreement with the theory.

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