Dynamic Range Vs Spectral Clarity Trade-off in All-Digital Frequency Synthesis via Single-Bit Sinewave Quantization

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***Abstract*—Single-bit Nyquist-rate quantization of sinewaves with amplitude dithering using a sequence of independent and uniformly distributed random variables has been proposed for all-digital RF frequency synthesis generation. This work demonstrates how we can improve the Dynamic Range using non-uniformly distributed dithering by selectively allowing some of the harmonics to be present in the spectrum (vs. the spurs-free output of the uniform distribution.). MATLAB simulation examples illustrate the results of the proposed approach.**

Keywords—Digital-to-frequency converter, direct digital synthesis, frequency spurs, quantization

I. Introduction

Over the past few years all-digital frequency synthesis has attracted the attention of the R.F.I.C. industry [1]-[3]. This was motivated by the increasing challenge in the design of R.F., analog and mixed-signal related circuit blocks, compared to alternative digital ones, as technologies downscale. Interestingly, efforts towards all-digital frequency synthesizers can be traced at least thirty years back [4]-[6].

Fully digital architectures have been proposed for generating synchronous single-bit digital outputs of sinewave-like spectrum which can be used as local-oscillator signals in RF chains and be amplified for transmission without distortion and with very high efficiency using a switching amplifier.

Here we consider a single-bit-output Nyquist-rate sinewave quantization scheme, discussed in [7]-[8], which is practically realized by a Direct Digital Synthesizer (DDS) with a 1-Bit output Nyquist-rate Digital to Analog Converter (DAC). Dithering is added to the output of the Look-Up-Table (LUT) before the single-bit quantization to alleviate the frequency spurs by breaking the periodicity of the truncation error.

It has been shown in [7]-[8] that uniform dithering of appropriate range completely eliminates the spurs. This paper illustrates that by selectively allowing some output harmonics we can reduce the noise floor level and improve the dynamic range of the output.

II. Abstract Architecture and Definitions

It is a common practice to use random dithering to suppress the frequency spurs of quantization in DDS [9]-[10] and data converters [11]. Here we consider the extreme case of a DDS with Nyquist-rate single-bit output quantization and amplitude dithering as shown in Figure 1. Sequence $\cos(\Omega k)$ can be generated by a phase accumulator and a LUT. The Zero Order Hold (ZOH) provides the continuous-time single-bit digital waveform output and all blocks are clocked by a reference clock of frequency $f_S = 1/T_S$.

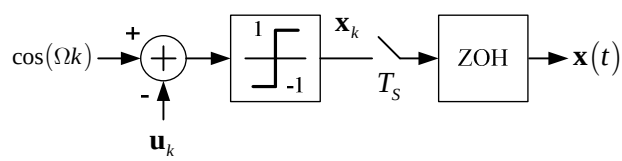

Figure 1: Dithered single-bit quantization of a sinewave

Let $\Omega = 2\pi w/q$ for some integers w and q such that $0 < w < q/2$. It is not necessary that q is a power of 2. Random dithering sequence $\{\mathbf{u}_k\}$ is subtracted from $\cos(\Omega k)$ resulting in the discrete-time single-bit (± 1) signal

$$\mathbf{x}_k = \operatorname{sgn}\left(\cos\left(2\pi wk/q\right) - \mathbf{u}_k\right). \tag{1}$$

We assume that $\{\mathbf{u}_k\}$ is composed of independent and identically distributed (IID) random variables of cumulative density function (CDF) $G:[-1,1] \to [0,1]$ with continuous second derivative. It is of course $\Pr(\mathbf{u}_k \le u) = G(u)$. Also, it is convenient for our analysis to express the CDF G as a series of Chebyshev polynomials of the first kind, i.e.,

$$G(u) = \frac{1}{2} + \frac{1}{2}\sum_{j=0}^{\infty} a_j T_j(u). \tag{2}$$

where the summand and multiplier $1/2$ are used to simplify the algebra. Coefficients a_j are derived according to [8], [12].

Inversely, CDF G can be defined using Eq. (2). Assuming the series in (2) converges appropriately and is term-by-term differentiable, G is a CDF if and only if $G(-1)=0$, $G(1)=1$ and $G'(u)\geq 0$ for every $u\in[-1,1]$. These three conditions can be expressed as

$$\sum_{j=0}^{\infty}(-1)^j a_j=-1 \quad , \quad \sum_{j=0}^{\infty}a_j=1 \tag{3}$$

and

$$\sum_{j=1}^{\infty} j a_j U_{j-1}(u)\geq 0 \qquad \forall u\in[-1,1] \tag{4}$$

using the fact that $T_j(\pm 1)=(\pm 1)^j$ for every $j=0,1,2,\ldots$, and $T_j'(u)=j\cdot U_{j-1}(u)$, for $j=1,2,3,\ldots$, where U_j is the j-th Chebyshev polynomial of the 2nd kind [12].

III. Output Spectrum

Under the assumption that the dithering sequence $\{\mathbf{u}_k\}$ is composed of IID random variables, the discrete-time sequence $\{\mathbf{x}_k\}$ is formed of independent but not identically distributed random variables. The distribution of $\mathbf{x}_k$ is a periodic function of $k\in\mathbb{Z}$ of period equal to q or a divisor of it. Therefore $\{\mathbf{x}_k\}$ is cyclostationary of period q. In this case it is common [13]-[14] to define the *period-average* autocorrelation function $\overline{r}_x(k)=\left(\sum_{m=0}^{q-1} r_x(k+m,m)\right)/q$, where $r_x(n,m)=E\{\mathbf{x}_n\mathbf{x}_m\}$; then, use $\overline{r}_x(k)$ to define the PSD of $\{\mathbf{x}_k\}$ as the discrete-time Fourier Transform of it, i.e., $s_x(\omega)=\sum_{k=-\infty}^{\infty}\overline{r}_x(k)e^{-ik\omega}$.

The ZOH in Figure 1 converts the discrete-time sequence $\{\mathbf{x}_k\}$ into the continuous-time signal

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty}\mathbf{x}_k\, p(t/T_S-k) \tag{5}$$

where $T_S=1/f_S$ is the sampling period and $p(t)$ is the pulse equal to 1 for $t\in[0,1)$ and zero otherwise corresponding to ZOH's operation. The PSD of the continuous-time signal $\mathbf{x}(t)$ can be expressed [8] using $s_x(\omega)$ as

$$S_x(f)=T_S\cdot \mathrm{sinc}^2(f\cdot T_S)\cdot s_x(2\pi f T_S). \tag{6}$$

The following Theorem, [8], based on the above definitions provides the means to calculate analytically the PSD $S_x(f)$.

<u>Theorem</u>: Based on the aforementioned definitions and assumptions, if $\gcd(w,q)=1$ then the PSD of $\mathbf{x}(t)$ is

$$S_x(f)=\mathrm{sinc}^2\left(\frac{f}{f_S}\right)\cdot\left(S_{HA}(f)+S_N(f)+S_{DC}(f)\right) \tag{7}$$

where the HArmonics, Noise and DC components are

$$S_{HA}(f)=\frac{1}{4}\sum_{h=1}^{\infty}b_h\left(\delta\left(f-\frac{h}{q}f_S\right)+\delta\left(f+\frac{h}{q}f_S\right)\right), \tag{8}$$

$$S_N(f)=\frac{1}{f_S}\cdot\left(1-a_0^2-\frac{1}{2}\sum_{j=1}^{\infty}a_j^2\right), \tag{9}$$

and

$$S_{DC}(f)=\frac{b_0+3a_0^2}{4}\delta(f) \tag{10}$$

respectively. The power of the frequency components at $\pm(h/q)f_S$, $h=0,1,2,\ldots$ in Eq. (8) is $b_h/4$ and

$$b_h \triangleq \sum_{r=-\infty}^{\infty} a_{I(h,r)}^2 \tag{11}$$

where $I(h,r)=|j_1 h+qr|$; constant j_1 (and k_1 which is not used in the expression) is derived solving the Diophantine equation $wj_1+qk_1=1$. Specifically, coefficient b_w of the frequency components at $\pm(w/q)f_S$ is $b_w=\sum_{r=-\infty}^{\infty}a_{|1+qr|}^2$. □

The application of the Theorem is illustrated in the following sections.

IV. DR and the Trade-Off With Spectral Clarity

In most practical cases the desirable frequency component is the one at $\pm(w/q)f_S$ with amplitude $\mathrm{sinc}^2(f/f_S)\cdot b_w/4$, captured by $S_{HA}(f)$ in Eq. (8). The output noise level has PSD $\mathrm{sinc}^2(f/f_S)\cdot S_N(f)$ captured by Eq. (9). We define the Dynamic Range (DR) as the ratio of the desirable signal's power to noise's PSD, $DR=10\log_{10}\left(\frac{\mathrm{sinc}^2(f/f_S)\cdot b_w/4}{\mathrm{sinc}^2(f/f_S)\cdot S_N(f)}\right)$, in (dB), which replacing the values of b_w and $S_N(f)$ becomes

$$DR=10\log_{10}\left(\frac{\sum_{r=-\infty}^{\infty}a_{|1+qr|}^2}{1-a_0^2-\sum_{j=1}^{\infty}\frac{a_j^2}{2}}\right)+10\log_{10}(f_S)-6.02 \quad \text{(dB)} \tag{12}$$

Note that the summand $10\log_{10}(f_S)$ in Eq. (12) should be expected because the power of the sinewave's quantization error is spread over the whole frequency bandwidth which is proportional to the sampling frequency.

A. Case 1: Uniformly Distributed CDF G

We consider first the case of dithering sequence $\{\mathbf{u}_k\}$ with *uniformly distributed* IID random variables, i.e. constant probability density $G'(u)=1/2$ in $[-1,1]$ and CDF given by $G(u)=(u+1)/2$. Observing Eq. (2) and using the fact that $T_1(u)=u$ we get $a_0=0$, $a_1=1$ and $a_k=0$ for $k=2,3,4,\ldots$.

Now we apply the Theorem for a pair of integers w and q with $0 < w < q/2$ and $\gcd(w,q)=1$. Since only $a_1 = 1$ is nonzero, for b_h, $h = 0,1,2,\ldots$ to be nonzero there must exist an integer $r \in \mathbb{Z}$ such that

$$I(h,r) = |j_1 h + qr| = 1 \tag{13}$$

Since (j_1, k_1) is a solution of the Diophantine equation $wj_1 + qk_1 = 1$ the general solution of Eq. (13) is

$$(h,r) = \pm(w,k_1) + \rho(q,-j_1),\ \rho \in \mathbb{Z} \tag{14}$$

and since $0 \le w < q$, the nonnegative values of h are $h = w + \rho q$ for $\rho = 0,1,2,\ldots$ and $h = -w + \rho q$ for $\rho = 1,2,\ldots$. We conclude that the only nonzero coefficients b_h, $h = 0,1,2,\ldots$ are b_w and $b_{\eta \cdot q \pm w}$ for $\eta = 1,2,3,\ldots$ giving

$$S_{HA}(f) = \frac{1}{4} \sum_{\substack{h=w,\ \eta \cdot q \pm w \\ \eta=1,2,3,\ldots}} \left(\delta\left(f - \frac{h}{q} f_S \right) + \delta\left(f + \frac{h}{q} f_S \right) \right),$$

$S_N(f) = 1/(2 f_S)$ and $S_{DC}(f) = 0$ respectively. So the only two frequency components in the frequency range $[0, f_S]$ are at $(w/q) f_S$ and its image at $(1 - w/q) f_S$. This is true in general for integers w and q satisfying our assumptions. Therefore uniform CDF G leads to spurs-free output.

For example if $w = 25$ and $q = 64$ then the only nonzero coefficients are b_{25} and $b_{\eta \cdot 64 \pm 25}$ for $\eta = 1,2,3,\ldots$. Moreover $b_{25} = b_{\eta \cdot 64 \pm 25} = 1$, $\eta = 1,2,3,\ldots$. The results are confirmed by the PSD in Figure 2 derived using simulation, where the weighting factor $\operatorname{sinc}^2(f T_S)$ is ignored.

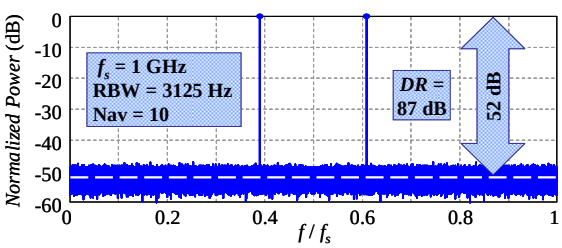

Figure 2: Spectrum of single-bit quantized sinewave with uniformly distributed dither when $w = 25$ and $q = 64$; the weighting factor $\operatorname{sinc}^2(f / f_S)$ is ignored. $f_S = 1\,GHz$, Resolution BW = 3125 Hz and waveform averaging Nav=10 runs.

Two more cases of simulated PSD when $a_0 = 0$, $a_1 = 1$ and $a_k = 0$ for $k = 2,3,4,\ldots$ are shown in Figures 3 and 4 for different values of w, q, f_s and waveform averaging runs Nav. As expected from the Theorem the spectra are spurs-free.

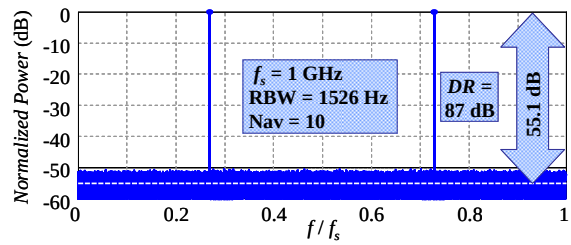

Figure 3: PSD of simulated random sequence $\{\mathbf{x}_k\}$ when $a_0 = 0$, $a_1 = 1$ and $a_k = 0$, $k = 2,3,4,\ldots$; $w = 17723$, $q = 2^{16}$; $f_S = 1\,GHz$, Resolution BW = 1526 Hz and waveform averaging Nav=10 runs.

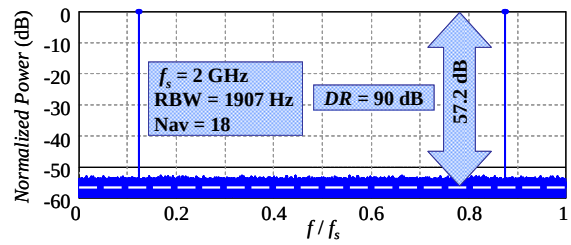

Figure 4: PSD of simulated random sequence $\{\mathbf{x}_k\}$ when $a_0 = 0$, $a_1 = 1$ and $a_k = 0$, $k = 2,3,4,\ldots$; $w = 2^{16} - 1$, $q = 2^{19}$; $f_S = 2\,GHz$, Resolution BW = 1907 Hz and waveform averaging Nav=18 runs.

The dynamic range for the case of $a_0 = 0$, $a_1 = 1$ and $a_k = 0$ for $k = 2,3,4,\ldots$, is derived from Eq. (12) to be

$$DR = 10\log_{10}(f_S) - 3.01 \text{ dB} \tag{15}$$

In the PSD of the case $w = 25$ and $q = 64$ in Figure 2 the dashed white line indicates the averaged noise floor power. For $f_S = 1\,GHz$ we get $DR \cong 87$ dB. Subtracting $10\log_{10}(RBW)$ dB, where $RBW = 3125\,Hz$, to account for the resolution BW used for the simulation we get a very good match to the simulated 52 dB (Figure 2). Similarly for the case $w = 17723$ and $q = 2^{16}$ in Figure 3 where again Eq. (15) gives $DR \cong 87$ dB and after subtracting $10\log_{10}(RBW) \cong 32$ we get about 55 dB (Figure 3). Similarly in the case in Figure 4.

B. Case 2: $a_1, a_3 \neq 0$ & All Other Coefficients a_k are Zero

Suppose now that $a_1, a_3 \neq 0$ and all other coefficients a_k are zero then since $T_1(u) = u$ and $T_3(u) = 4u^3 - 3u$ we have $G(u) = \frac{1}{2} + \frac{a_1}{2}u + \frac{a_3}{2}(4u^3 - 3u)$ from Eq. (2). For G to be a CDF Eqs. (3) and (4) must hold implying $a_1 + a_3 = 1$ and $2G'(u) = a_1 + a_3(12u^2 - 3) \ge 0$ for every $u \in [-1,1]$.

Note that function $2G'(u)$ achieves its minimum either at $u = 0$ or at $u = \pm 1$ so $G'(u) \ge 0$ for every $u \in [-1,1]$ if and only if $a_1 - 3a_3 \ge 0$ and $a_1 + 9a_3 \ge 0$. Therefore G is a CDF if and only if $a_1 + a_3 = 1$, $a_1 + 9a_3 \ge 0$ and $a_1 - 3a_3 \ge 0$. The solution is $a_1 = (6 + 3\rho)/8$ and $a_3 = (2 - 3\rho)/8$, $\rho \in [0,1]$.

Assuming $q > 4$ implies $\sum_{r=-\infty}^{\infty} a_{|1+qr|}^2 = a_1^2$ and Eq. (12) gives $DR = 10\log_{10}\left(\frac{2a_1^2}{2-\left(a_1^2+a_3^2\right)}\right) + 10\log_{10}\left(f_S\right) - 6.02$ dB. Using the expressions of a_1 and a_3 above, DR becomes a strictly increasing function of ρ with maximum value $DR = 10\log_{10}\left(f_S\right) - 0.55$ (dB) for $\rho = 1$, corresponding to $a_1 = 9/8$ and $a_3 = -1/8$. In this case the DR is about 2.5 dB higher than in Case I but the 3rd harmonic is present here as shown in the following.

For $w = 25$ and $q = 64$ the simulated PSD is shown in Figure 5. Applying the Theorem again we conclude that the only frequencies in the spectrum $\left(0, f_S\right)$ are at $\left(25/64\right) f_S$ and $\left(1-25/64\right) f_S$ corresponding to the fundamental and its image as well as $\left(-1+3\cdot 25/64\right) f_S$ and $\left(2-3\cdot 25/64\right) f_S$ corresponding to the 3rd harmonic and its image. Moreover for $h = 0,1,2,...,63$ the only nonzero b_h are $b_{25} = (9/8)^2$, $b_{39} = (9/8)^2$, $b_{11} = (1/8)^2$ and $b_{53} = (1/8)^2$ corresponding to frequencies above. Results agree with simulation in Figure 5.

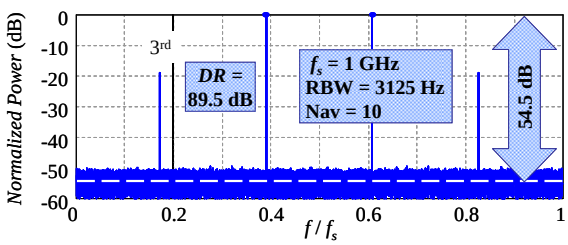

Figure 5: PSD of simulated random sequence $\{\mathbf{x}_k\}$ when $a_1 = 9/8$, $a_3 = -1/8$ and all other coefficients $a_k = 0$; $w = 25$ and $q = 64$

C. *Case 3: $a_1, a_3, a_5 \neq 0$ & All Other Coefficients a_k are Zero*

The case of $a_1, a_3, a_5 \neq 0$ with all other coefficients a_k zero implies $G(u) = \frac{1}{2} + \frac{a_1}{2}T_1(u) + \frac{a_3}{2}T_3(u) + \frac{a_5}{2}T_5(u)$ which is a CDF if and only if $a_1 + a_3 + a_5 = 1$ and $2G'(u) = 80a_5u^4 + \left(12a_3 - 60a_5\right)u^2 + a_1 - 3a_3 + 5a_5 \geq 0$ for all $u \in [-1,1]$.

The derivation of $\left(a_1, a_3, a_5\right)$ maximizing DR is more involved than before. It turns out that the maximum is achieved for $a_1 = 1.1906$, $a_3 = -0.2375$, $a_5 = 0.0469$ and all other coefficients a_k zero; $DR = 10\log_{10}\left(f_S\right) + 1.31$ (dB), i.e. about 4.3 dB higher than with uniformly distributed dither in Case I.

For $w = 25$ and $q = 64$ the simulated PSD is shown in Figure 6. The fundamental the 3rd and the 5th harmonics and their images are the only components in the spectrum, which can be shown using the Theorem as well.

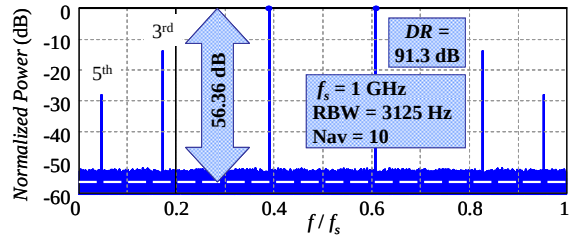

Figure 6: PSD of simulated random sequence $\{\mathbf{x}_k\}$ when $a_1 = 1.1906$, $a_3 = -0.2375$, $a_5 = 0.0469$, all other coefficients $a_k = 0$; $w = 25$, $q = 64$

V. Conclusions

Single-bit, Nyquist-rate quantization of sinewave with additive random dithering formed of independent and identically distributed random variables has been discussed as a means for all-digital frequency synthesis. Using the uniformly distributed dither, resulting in spurious free output, as a reference, other distributions have been considered in an effort to increase the dynamic range by trading-off the presence of selected harmonics in the output for lower output noise floor. An improvement of about 4.3 dB has been shown when the third and fifth harmonics are allowed to be present. Examples based on MATLAB simulation have been discussed illustrating the presented theory.

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