Continuous-Time Signal Processing with Time-Variant Delay

Paul P. Sotiriadis
Sotekco Electronics LLC
Baltimore, MD, USA
pps at ieee dot org

Robert Adams
Analog Devices Inc.
Wilmington, MA
USA

***Abstract*—This work concerns with the behavior and characterization of continuous-time signal processing structures involving time-variant delays. Both loop-less and looped structures are considered. Their cascades, invertibility and uniqueness of representation are studied along with their similarity transformations leading to the concept of Externally Time-Invariant Internally Time-Variant delay structures.**

I. Introduction

High-speed continuous-time signal processing has been revitalized over the past decade due to capabilities modern semiconductor technologies offer as well as new approaches to circuit signal processing [1-3].

Delay, as a signal processing operation, is fundamental in DSP but has received rather limited attention in continuous-time signal processing. Although it usually appears as a parasitic behavior in signal transmissions and in feedback loops, delay operation is used on purpose in some important analog applications.

In the RF, microwave and photonics regime, delay operation is implemented using a piece of transmission line, e.g. a micro-strip, a waveguide, or even a coaxial cable, and it is critical in applications like beam-forming in radars, RAKE receivers [4-5], as well as in instrumentation technology, e.g. in phase noise measurements.

These applications use delay in a *forward* manner without feedback or loop architectures involved. Certain *comb* filters that also have important applications in microwave signal generation; optical filters etc. also have forward delay structures. However, there are certain comb filters and optical devices based on cavities or optical fibers that use delay (purposely) in closed loops [6]. An important use of it is in very low phase noise oscillators.

In the aforementioned applications, the delays are either fixed or, slowly adjustable which can be treated as fixed ones with respect to the frequency (range) of the signals involved. Naturally, one would wonder whether time-variant delays could offer any further advantage.

This work concerns with the behavior and characterization of continuous-time signal processing structures involving one or more controllable time-delays. We study loop-less and looped structures and derive expressions for the behavior of their cascades, conditions for the invertibility of their operation, properties of their representations, and properties of their similarity and change-of-time-variable transformations.

Due to space limitations, most of the results are stated without their mathematical proofs, some of which are lengthy.

II. Definitions and Basic Properties

A. Backward and Forward Delay

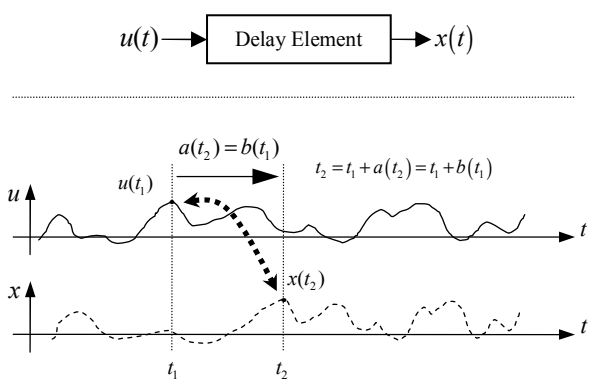

Figure 1: Input-Output Backward and Forward Delay

The time delay operation can be interpreted in at least two different ways. The *Backward* and the *Forward* ones are considered here. In the case of *backward* delay, the value of the output x at time t_2 is that of the input u, $a(t_2)$ seconds before as shown in Figure 1, i.e. we look *backward* in time. Therefore in this case we have $x(t)=u(t-a(t))$.

Now consider the value $u(t_1)$ of the input signal entering the delay element at time t_1. We can *arrange* so that $u(t_1)$ appears at the output $b(t_1)$ seconds later as shown in Figure 1. In this case we have that $u(t_1)=x(t_2)=x(t_1+b(t_1))$ and so it is $u(t)=x(t+b(t))$.

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Introducing function c such that $c(t) = t + b(t)$, we observe that if it is invertible[1] then the forward and backward delays are equivalent if and only if $a(t) = t - c^{-1}(t)$ for all considered values of t.

B. Notation and Assumptions

From now on all delays are considered to be *backward* ones. The notation used is shown in Figure 2 below.

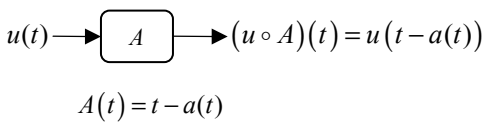

Figure 2 : Notation of the backward delay

The delay operation is considered as a *composition* of mathematical functions, i.e. the output is the function $u \circ A$ where function A is defined as $A(t) = t - a(t)$.

Certain assumptions are in order for the time-delay function $a(t)$. First, it makes sense to assume that $a(t) \geq 0$ for all $t \in \mathbf{R}$. This implies $A(t) \leq t$ and therefore A is indeed a delay operation. (Note that in certain cases the domain of functions a and A may be a proper subset of $\mathbf{R}$.) We may also want to assume that function a, and so A, has some smoothness properties, e.g., its first derivative exists and is continuous.

The inverse function $A^{-1} : \mathbf{R} \to \mathbf{R}$ is required in many instances. A sufficient condition for it to exist is that $\frac{dA}{dt}(t) = 1 - a'(t) > 0$, or equivalently that $a'(t) < 1$, which implies that A is a strictly increasing function of time.

C. Cascades

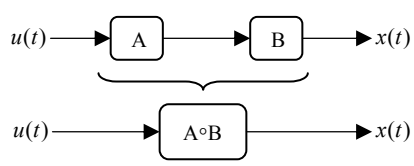

Figure 3 : Equivalent Delay of a Cascade of Delays

In contrast to the transfer-function notation in L.T.I. systems, in a cascade of delays, the last delay operation is the rightmost one in the function composition, i.e.,

$$x(t) = u\left(A\left(B(t)\right)\right) = u\left(t - b(t) - a\left(t - b(t)\right)\right).$$

The following properties of function composition are useful in dealing with the compositions of delay functions:

1. In general $A \circ B \neq B \circ A$,
2. $A \circ (B \circ C) = (A \circ B) \circ C$,
3. $F(A_1, A_2, \ldots, A_n) \circ B = F(A_1 \circ B, A_2 \circ B, \ldots, A_n \circ B)$.

[1] Mathematical formalities are ignored for presentation simplicity.

Finally, we denote the kth self-composition of A, as $A \circ A \circ \ldots \circ A = A^{\circ k}$ and we allow k to take *negative* integer values, as well, if function A^{-1} exists. Then for $k < 0$ we define $A^{\circ k} = \left(A^{-1}\right)^{\circ |k|}$.

D. Delay and Gain do Not Commute – In general

One has to be careful and not apply transformations used in L.T.I. systems automatically. Consider the two block diagrams below involving a delay and a time-variant gain stage in reverse orders. Note that only in the second case the delay impacts the gain factor.

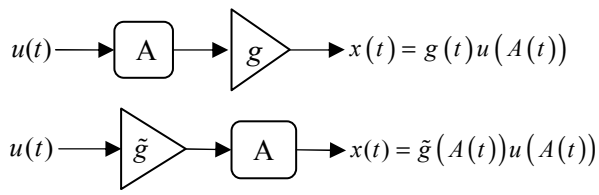

Figure 4 : Delay and Gain stages in a Cascade

E. Delay-Loops Without Gain.

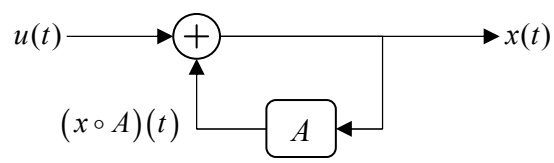

Figure 5 : Gain-Less Delay Loop

Consider the loop in Figure 5. It implies the functional algebraic equation $x(t) = u(t) + (x \circ A)(t)$ which can be used repeatedly to yield $x \circ A^{\circ(k-1)} = u \circ A^{\circ(k-1)} + x \circ A^{\circ k}$ (here we have dropped the dependence on t for convenience). Summing over k and assuming[2] that $a(t) \geq a_{\min} > 0$ for all $t > 0$, and that $x(t) = 0$ for all $t < T$, we conclude that

$$x(t) = \sum_{k=0}^{\infty} \left(u \circ A^{\circ k}\right)(t).$$

Moreover, there exists a positive-integer valued function $K(t)$ such that

$$x(t) = \sum_{k=0}^{K(t)} \left(u \circ A^{\circ k}\right)(t)$$

This expression of $x(t)$ as a finite sum is helpful in the numerical treatment of the signals involved in time-variant delay structures (more assumptions are needed of course).

[2] There are many alternative assumptions leading to the same expression for $x(t)$. All proofs are omitted due to length limitations of the paper.

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F. Delay-Loops With Gain

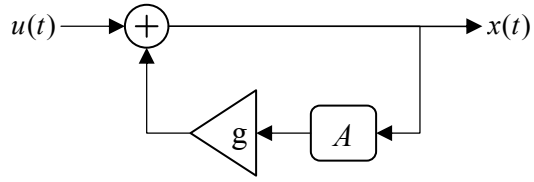

Figure 6 : Gain Delay Loop

Now we add a linear, time-variant gain stage in the loop after the delay. The loop implies the functional equation $x(t)=u(t)+g(t)(x\circ A)(t)$. (Note that both boxes g and A require a reference of time t). By making assumptions on A and x similar to those in Section E, we can conclude that

$$x(t)=u(t)+\sum_{k=1}^{\infty}\left(\prod_{j=0}^{k-1}g\left(A^{\circ j}(t)\right)\right)u\left(A^{\circ k}(t)\right)$$

as well as that there exists a positive-integer valued function $K(t)$ such that

$$x(t)=u(t)+\sum_{k=1}^{K(t)}\left(\prod_{j=0}^{k-1}g\left(A^{\circ j}(t)\right)\right)u\left(A^{\circ k}(t)\right)$$

G. Inversion of the Loop's Operation

The inversion of the operation of the gain-delay, or delay only, loop becomes obvious if we rewrite the loop's equation as $u=x-g(x\circ A)$. The process is shown below.

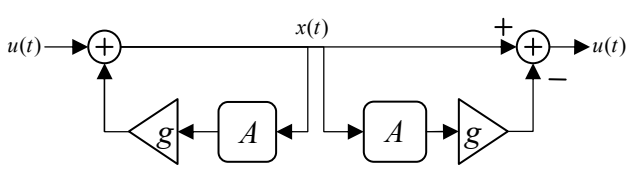

Figure 7 : Inversion of the Gain Delay Loop's Operation

H. Cascade of Gain Delay Loops

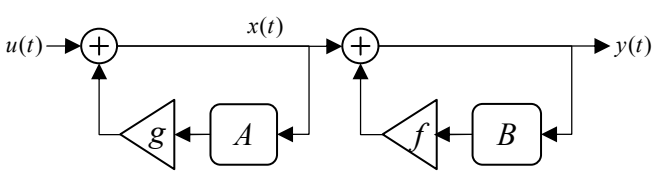

Figure 8 : Cascade of Gain Delay Loops

The cascade of two gain-delay loops leads to the relatively complex expression for the output y,

$$y=u+\sum_{k=1}^{\infty}\left(\left(u\circ A^{\circ k}\right)\prod_{j=0}^{k-1}g\circ A^{\circ j}+\left(u\circ B^{\circ k}\right)\prod_{i=0}^{k-1}f\circ B^{\circ i}\right)$$
$$+\sum_{m,k=1}^{\infty}\left(\prod_{i=0}^{m-1}f\circ B^{\circ i}\right)\left(\prod_{j=0}^{k-1}g\circ A^{\circ j}\circ B^{\circ m}\right)\left(u\circ A^{\circ k}\circ B^{\circ m}\right) \quad (1)$$

One way to make the algebra more compact is by expressing operators A,B using *infinite* vectors and matrices.

III. Block Diagram Transformations

Here we discuss some basic transformations of structures involving time-variant delays. We consider first the case of delay-only loops.

A. Transformations of Loops without Gain

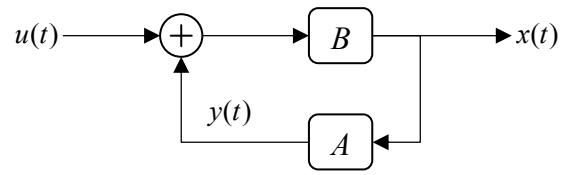

Figure 9 : Gain-Less Loop with Delays in the Forward and Feedback Paths

The loop implies the two equations: $y(t)=(x\circ A)(t)$ and $x(t)=((u+y)\circ B)(t)=(u\circ B)(t)+(y\circ B)(t)$ which combined give $x=u\circ B+x\circ A\circ B$. This implies directly the input-output equivalence between the loops in Figure 9 and in Figure 10 where B has been shifted counter-clock-wise.

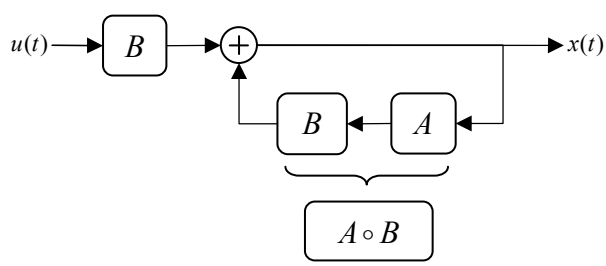

Figure 10 : Equivalent Gain-Less Loop with Input and Feedback Delays

Both loops in Figure 9 and Figure 10 have response

$$x(t)=\sum_{k=0}^{\infty}\left(u\circ B\circ\left(A\circ B\right)^{\circ k}\right)(t).$$

A third equivalent block-diagram is shown in Figure 11. There, delay B has been shifted clock-wise. The proof of equivalence is based on identity $(B\circ A)^{\circ k}\circ B=B\circ(A\circ B)^{\circ k}$.

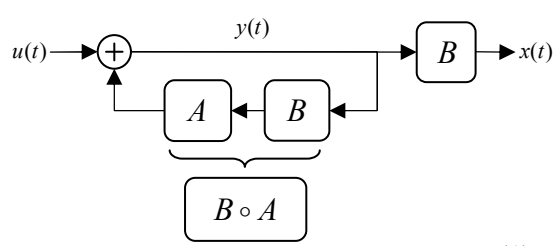

Figure 11 : 3rd Equivalent Transformation–Note the reversed order of $B\circ A$.

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The equivalence of the three block diagrams allows us to manipulate and simplify more complex structures with delays. The above results extend to the case of gain-delay loop below.

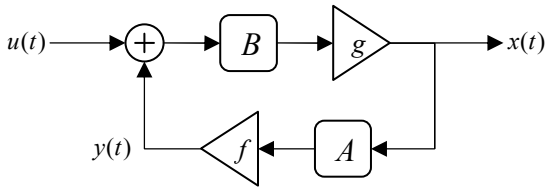

Figure 12 : Gain-Delay Loop with Forward and Feedback Gain-Delays

Here however, attention must be paid to the order in which one shifts the delays and the gains.

IV. Change of Time-Variables and the Time-Based Similarity Transformation $B^{-1} \circ A \circ B$

The change of variables here has the meaning of changing the time reference, i.e. in a sense it is a time modulation. Consider the loop in Figure 6. The function of time x can be called the *state*[3] of the loop because the complete behavior of the loop is fully determined by x. Moreover, the loop in Figure 6 implies $x = u + g \cdot x \circ A$.

Now let $B : \mathbf{R} \to \mathbf{R}$ be a delay function and let's define a new state (function) z such that $z = x \circ B$. Composing the loop equation with B gives $x \circ B = u \circ B + (g \circ B) \cdot (x \circ A \circ B)$. If in addition we assume that B is invertible we also get that $z = u \circ B + (g \circ B) \cdot (z \circ B^{-1} \circ A \circ B)$ which is realized by the loop below.

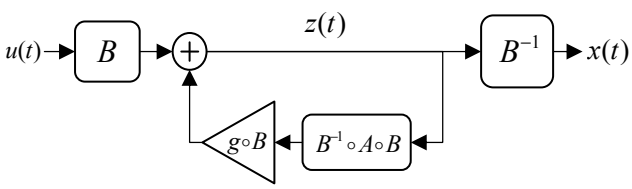

Figure 13 : *Similarity* Transformation in the Time Domain

It can be shown that $B^{-1} \circ A \circ B$ is a delay function if and only if $h(t) \geq 0$ for all t, where $h = b + a \circ B - b \circ B^{-1} \circ A \circ B$. A sufficient condition for the last inequality to be valid is

$$\min_{t \in \mathbf{R}} b(t) + \min_{t \in \mathbf{R}} a(t) \geq \max_{t \in \mathbf{R}} b(t).$$

V. Application: Externally Time-Invariant Internally Time-Variant Operations

An interesting question from both theoretical and practical perspective is: *When is a structure containing time-variant, or controllable delays, input-output time-invariant ?*

[3] whether the term *state variable* is accurate is subject to discussion because both the instantaneous time value $x(t)$ and past values $x(t-\sigma)$ are needed to determine the behavior of the loop

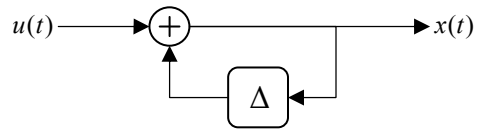

Figure 14 : Time-Invariant Integrator

Consider the *integrator* in Figure 14 where $\Delta(t) = t - T$. The loop implies the equation $x = u + x \circ \Delta$. Let B be an invertible delay function and perform the similarity transformation of the previous section. It results in Figure 15.

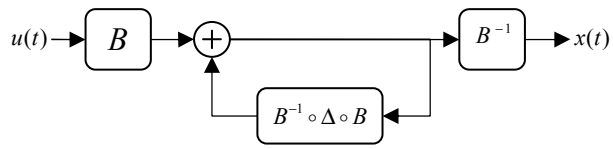

Figure 15 : Transformed Integrator

Following the results of the previous section, we can assume that $T \geq \max_{t \in \mathbf{R}} b(t) - \min_{t \in \mathbf{R}} b(t)$ in order to guarantee that $B^{-1} \circ \Delta \circ B$ is a delay function. Moreover, by adding one more fixed delay, $\tilde{\Delta}(t) = t - \sigma$, at the output after B^{-1} (not shown in the figure) with the property that $\sigma \geq \max_{t \in \mathbf{R}} b(t)$, we also guarantee that the pair $B^{-1} \circ \tilde{\Delta}$ is a delay function and therefore physically realizable.

Based on the previous analysis, the structure in Figure 15 is the most general class of (gain-less) time-variant delay (single) loops that have input-output behavior identical to that of the time invariant integrator in Figure 14.

VI. Conclusions

Continuous-time signal processing structures containing time-variant delays have been analyzed theoretically. This has led to the derivation of several properties of loop-less and looped structures and their cascades and similarity transformations, as well as to the definition of the class of externally time-invariant, internally time-variant delay structures.

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