Information Storage Capacity of Crossbar Switching Networks

Paul P. Sotiriadis
Department of Electrical and Computer Engineering
Johns Hopkins University
Baltimore, MD
pps@jhu.edu

ABSTRACT

In this work we ask the fundamental question: ***How many bits of information can be stored in a crossbar switching network?*** The answer is trivial when the switches of the network are in series with diodes (semi-conductive) but it is complicated when the switches are regular contacts. Exact explicit expressions and simple asymptotic bounds of the storage capacity (in bits) are derived for the general crossbar switching network with regular contact switches.

Categories and Subject Descriptors

B.3 [**Memory Structures**]: Misc.; H.0 [**Information Systems**]

General Terms

Design, Measurement, Performance, Theory.

Keywords

Capacity, Crossbar, Device, Information, Memory, Network, Nanotechnology, Nanotube, Nanowire, Storage, Switching.

1. INTRODUCTION

Crossbar switching networks (CSNs) have been used extensively in communication and computation systems. Recent advances in nanotechnology have enabled the construction of crossbar switching structures using nano-sized wires [1], [2]. The small size and high density of these structures have motivated the study of their possible applications as high density interconnect, computation and information storage devices [3]-[8].

The storage capacity of crossbar switching networks, Figure 1, is derived in this paper. Both exact explicit expressions and simple asymptotic bounds are given. A more compact exact expression as well as an asymptotic expression of the storage capacity can be found in a follow up work [9].

We consider general crossbar switching networks whose switches between each horizontal and each vertical wire are regular contacts with two possible states, one of high and one of low resistance. This is in contrast to crossbar switching networks whose switches have semi-conductive properties as that shown in Figure 2.

GLSVLSI'03, April 28-29, 2003, Washington, DC, USA.

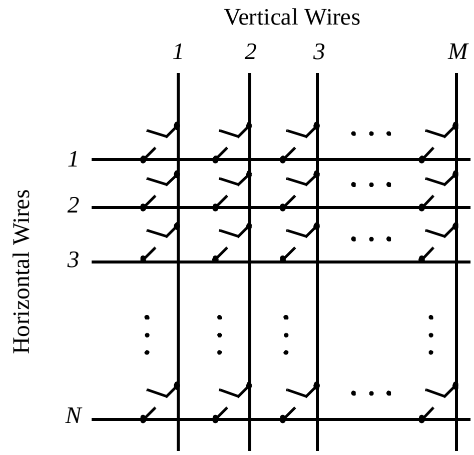

Figure 1. $N \times M$ **crossbar switching network**

2. INFORMATION STORAGE IN CSNS

In order to proceed in estimating the information capacity of a general $N \times M$ CSN like that of Figure 1, we make two assumptions that should be considered reasonable for every practically useful implementation. First, the $N \times M$ switches are assumed to have a high ratio of r_{off}/r_{on} resistance and can be switched *on* and *off* independently of each other. Second, the ratio of r_{off} over the total resistance of a wire (or nanotube) is assumed to be relatively high. In the analysis that follows these assumptions allow us to think of CSNs as being composed out of ideal switches and ideal wires.

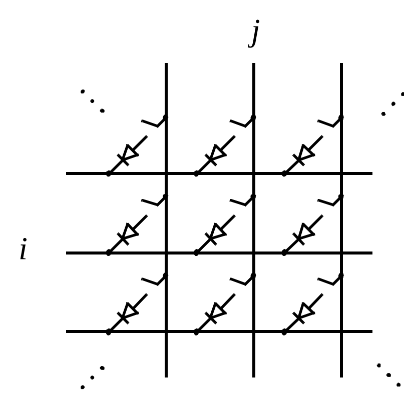

Figure 2. Switches in series with (ideal) diodes

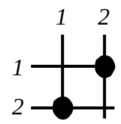

Figure 3. Notational convention: closed switches are shown as dots

With a pair (i, j) of wires we mean the pair of the $i-th$ horizontal and $j-th$ vertical wire. The pair (i, j) also corresponds to the position of the switch between those two wires. For pictorial simplification we put a black *dot* at the intersection of the pair (i, j) to represent a closed switch. The convention is shown in Figure 3 below.

2.1 Switches with and without diodes

Suppose for the moment that all switches were in series with ideal diodes (the cathodes are connected to horizontal wires), exactly as in Figure 2. Moreover, suppose that the $N \times M$ switches have a given *on-off* configuration. In order to extract the particular configuration we can do a series of $N \times M$ experiments: For every pair (i, j), $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$ we "measure" the current $I_{i,j}$, see Figure 4. If $I_{i,j}$ is non-negligible (recall the two assumptions we made) we conclude that the switch (i, j) is closed.

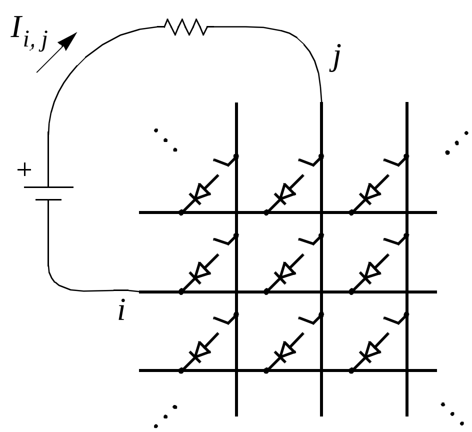

Figure 4. Extracting the configuration of a CSN that has semi-conductive switches.

The important point here is that the $N \times M$ connections are *independent* in the sense that the measured current, Figure 4, depends on the state of switch (i, j) and it is independent of the state of any other switch. Whereas, in the case of the CSN of Figure 1 this is not true. For example, performing the same experiment for the pair $(1, 2)$ in the two different configurations of Figure 5 we get the same result.

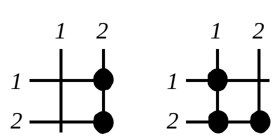

Figure 5. Two configurations of a 2×2 CSN.

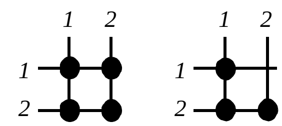

Figure 6. Two *different* configurations of a 2×2 CSN having the *same* conductivity proprties.

Therefore in the case of CSNs *without* diodes, the results of the different experiments are neither independent nor they correspond to unique states of the particular switches.

Now, suppose that instead of examining individual pairs of wires we consider the set of all $N \times M$ experiments as a whole. By doing so we conclude that the pairs of connected wires in the left configuration of Figure 5 are: $(1, 2)$ and $(2, 2)$ while the pairs of connected wires in the right configuration are: $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 2)$. This allows us to distinguish the two configurations of Figure 5. The same is not true though for the two configurations in Figure 6.

Note that the configurations of Figure 6 can*not* be distinguished based on the states of their switches (i.e. the left configuration has all switches closed while the right one has only three one them closed.) This is because the switches are *not* directly accessible to the outside of the CSN, the switches are *hidden* inside the CSN, Figure 7. In this sense, the two configurations above have to be considered as being *identical*.

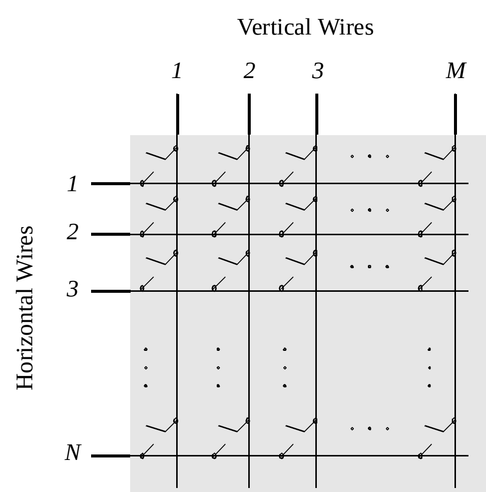

Figure 7. The CSN seen as an $N + M$ terminal device

The above examples and discussion can be directly extended to the case of the general $N \times M$ CSN. Therefore we are forced to think of its different configurations as $(N + M)$*-terminal electrical devices* because it is their (total) electrical behavior that differentiates them and *not* the setup of the switches.

Note that in the case of the switching network of Figure 2 there are exactly $2^{N \times M}$ possible electrical behaviors, or equivalently electrical devices, that can be realized by the different configurations of the switches. In this sense we can say that the information capacity of the network is $N \times M = \log 2^{N \times M}$ bits (throughout the paper $\log$ stands for the logarithm base 2). Similarly, suppose that the network of Figure 1 can realize $\boldsymbol{D}$ electrical devices by the different configurations of its switches. Then, its storage capacity is $\boldsymbol{B} = \log \boldsymbol{D}$.

In order to proceed in the calculation of $\boldsymbol{B}$ it is important to give some more formal definitions, this is done in the following section.

3. THE NUMBER OF DEVICES

Definition 1: *A device D is a CSN of Figure 1 along with a configuration of its switches.*

To simplify our discussion we use the subscript h for the horizontal wires and the subscript v for the vertical wires of the CSN. Then the set of all wires is:

$$T = \{1_h, 2_h, \ldots, N_h, 1_v, 2_v, \ldots, M_v\}.$$

Following our discussion in Section 2, every device is uniquely characterized by the way its wires are connected to each other. In other words, the device is uniquely characterized by the *partition* of the set T into maximal subsets of wires that are electrically connected. Such a partition will be called the *connectivity partition* of the device.

Example 1: The connectivity partitions of the devices in Figure 5 are, $P_{D_1} = \{\{1_h, 2_h, 2_v\}, \{1_v\}\}$ and $P_{D_1} = \{\{1_h, 2_h, 1_v, 2_v\}\}$ respectively. The first one has two *blocks* (i.e. connected components) while the second has only one.

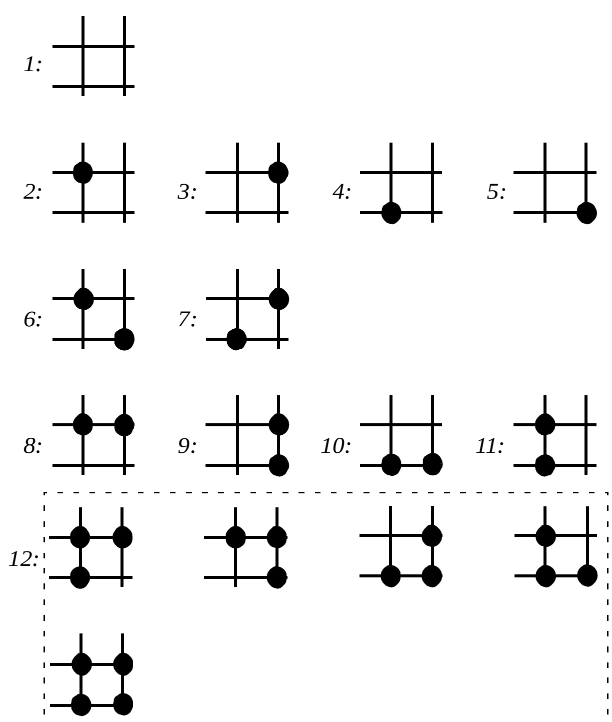

Figure 8. The 16 configurations of a 2×2 CSN and the 12 devices they realize.

Definition 2: *Two devices D_1, D_2 are identical if and only if they have the same connectivity partition.*

Example 2: In Figure 8 we see the sixteen configurations of the switches of a 2×2 CSN. There are *only* 12 devices (i.e. electrical behaviors) that are realized. The last 5 configuration realize the same device since all wires are connected together.

A simple asymptotic lower bound of the number of bits $\boldsymbol{B}$ that can be stored in an $N \times N$ CSN is derived in the following lemma.

Lemma 1: *The information capacity, $\boldsymbol{B}$, of an $N \times N$, CSN is asymptotically equal to or greater than $N \cdot log_2 N$ bits as $N \to \infty$.*

Proof: There are $N!$ configurations of the switches such that every horizontal wire is connected to exactly one vertical wire. The situation is shown in Figure 9.

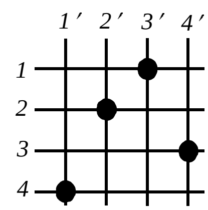

Figure 9. 1-1 connection of the wires

For every N there is some θ_n, with $1 < \theta_n < e$, such that [10]:

$$N! = \sqrt{2\pi} \cdot (N+1)^{N+1/2} \cdot e^{-(N+1)} \cdot \theta_n.$$

Therefore we have $\frac{\log N!}{N \cdot \log N} \to 1$ as $N \to \infty$. □

The following theorem provides an exact expression of the information capacity of an $N \times M$ CSN. The expression involves the *Stirling numbers of the second kind* $S(n, m)$. By its definition $S(n, m)$ is the number of distinct ways that a set of n elements can be partitioned into m non-empty subsets [10]. A convenient expression of $S(n, m)$ is given by (1), see for example [11].

$$S(n, m) = \frac{1}{m!} \sum_{t=0}^{m} (-1)^{m-t} \cdot \binom{m}{t} t^n \tag{1}$$

Theorem 1: *The information capacity of an $N \times M$ CSN is $\boldsymbol{B} = \log \boldsymbol{D}$ where $\boldsymbol{D}$ is given by expression (2) and $S(n, m)$ is the Stirling number of the second kind.*

$$\boldsymbol{D} = 1 + \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \binom{N}{i}\binom{M}{j} \sum_{q=1}^{min\{N-i, M-j\}} S(N-i, q) S(M-j, q) q! \tag{2}$$

Instead of enumerating the devices using a brute force, which is not an easy task, we use the following standard trick of combinatorics to prove the theorem. We construct a bijective map between the set of devices and another set whose elements we can enumerated easier. This is done in the following lemma. Note that the sets of the horizontal and the vertical wires of an $N \times M$ CSN are denoted by $\boldsymbol{H} = \{1_h, 2_h, \ldots, N_h\}$ and $\boldsymbol{V} = \{1_v, 2_v, \ldots, M_v\}$ respectively. Moreover for any two sets A, B, $A \subset B$ includes the case $A = B$ and $|A|$ is the number of elements in the set A.

Lemma 2: *The set of connectivity partitions of an $N \times M$ CSN can be mapped bijectively into the set of six-tuples (H, V, q, P, R, f) that (are defined below and) have the following properties:*

1. *$H \subset \boldsymbol{H}$, it may be $H = \varnothing$.*
2. *$V \subset \boldsymbol{V}$, it may be $V = \varnothing$.*
3. *q : nonnegative integer s.t. $0 \le q \le min\{N - |H|, M - |V|\}$.*
4. *If $q > 0$, P is a partition of $\boldsymbol{H} - H$ into q non-empty subsets otherwise $P = \varnothing$.*
5. *If $q > 0$, R is a partition of $\boldsymbol{V} - V$ into q non-empty subsets, otherwise $R = \varnothing$.*
6. *A bijective map f from P to R. (if $q = 0$ then f is taken to be the dummy function f_d:$\{0\} \to \{0\}$*

Proof of lemma 2: Given a device D let $P_D = \{T_1, T_2, \ldots, T_r\}$ be its connectivity partition for some positive integer r. Also let H be the set of horizontal wires that are *not* connected to any vertical wire. Similarly, let V be the set of vertical wires that are *not* connected to any horizontal wire. We set, $i = |H|$, $j = |V|$ and $q = r - i - j$. (It may be $i = 0$ or $j = 0$).

Without loss of generality we can assume that all elements in H and V are contained in the last $r - q = i + j$ blocks of P_D, i.e. $H \cup V = T_{q+1} \cup \ldots \cup T_r$. (Note that in this case T_{q+1}, T_{q+2},..., T_r are singletons.) Also note that each of the remaining subsets T_1, T_2,..., T_q must contain at least one horizontal and one vertical wire. For every $k = 1, 2, \ldots, q$ we can decompose T_k into $T_k = H_k \cup V_k$ where $H_k \subset \boldsymbol{H}$ and $V_k \subset \boldsymbol{V}$. We define P, R to be the partitions $P = \{H_1, H_2, \ldots, H_q\}$, $R = \{V_1, V_2, \ldots, V_q\}$ of the sets $\boldsymbol{H} - H$ and $\boldsymbol{V} - V$ respectively. Note that P and R are connectivity partitions.

Up to here we have shown that P_D implies the five-tuple (H, V, q, P, R). The six-tuple is completed by the mapping f from P to R such that $f(H_k) = V_k$ for every $k = 1, 2, \ldots, q$, assuming that $q > 0$. If $q = 0$ we set f to be the dummy function f_d defined in property 6 of the lemma.

To go from a six-tuple (H, V, q, P, R, f) to a connectivity partition P_D, and so to the corresponding device D, we switch-on all switches (a, b) such that a belongs to some S, $S \in P$ and b belongs to $f(S)$. We switch-off all the remaining switches.

Finally, it is straight forward to verify that the mapping between connectivity partitions and the set of six-tuples is bijective. □

Proof of theorem 1: There is the trivial case $H = \boldsymbol{H}$ and $V = \boldsymbol{V}$ where no wire is connected to any other wire. (Note that if one of these equalities holds then the other one must hold as well because it is impossible to have all horizontal wires disconnected while some of the vertical ones are connected to each other and via versa.) We count 1 for the above case and exclude it in the following analysis.

Given some $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, M-1$ we choose the sets of disconnected horizontal and vertical wires $H \subset \boldsymbol{H}$ and $V \subset \boldsymbol{V}$ such that $|H| = i$ and $|V| = j$. There are $\binom{N}{i}\binom{M}{j}$ ways to do so. For $q = 1, 2, \ldots, min\{N - |H|, M - |V|\}$, we choose partitions P and R of $\boldsymbol{H} - H$ and $\boldsymbol{V} - V$ respectively, each having exactly q blocks. By the definition of the second Stirling numbers there are $S(N-i, q)S(M-j, q)$ ways of doing so. Finally, for every partition pair (P, R) there are $q!$ distinct bijective mappings from P to R. Summarizing the above we get expression (2). □

Example 3: Figure 10 presents the graph of the capacity $\boldsymbol{B}$ of $N \times N$ CSNs with N ranging from one to one hundred along with the two asymptotic bounds $N\log N$ and $2N\log N$ of lemma 1 and corollary 1 (see next page) respectively.

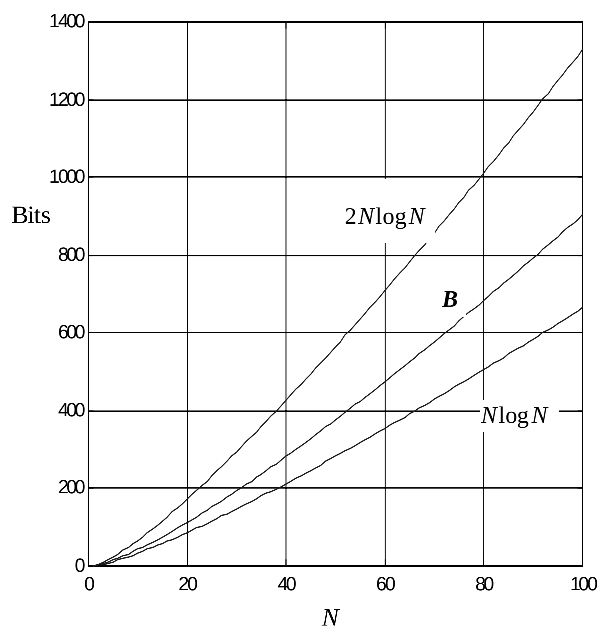

Figure 10: Information Capacity $\boldsymbol{B}$ of $N \times N$ CSNs with $N = 1, 2, \ldots, 100$.

There are alternative expressions of the number of devices $\boldsymbol{D}$. One of them is given in lemma 3 and is used in the derivation of an asymptotic upper bound of $\boldsymbol{B}$ in corollary 1 below.

Using more involved mathematical analysis it is possible to get other more compact expressions of $\boldsymbol{B}$. A few can be found in [9] along with a simple asymptotic expression.

Lemma 3: *The number of devices* $\boldsymbol{D}$ *can be expressed as:*

$$\boldsymbol{D} = 1 + \sum_{q=1}^{min\{N,M\}} \sum_{i=0}^{N-q} \sum_{j=0}^{M-q} \binom{N}{i}\binom{M}{j} S(N-i,q)S(M-j,q)q! \quad (3)$$

Proof: The proof is similar to that of theorem 1. Here we count first with respect to the number q of the blocks in the partitions of the sets of the connected wires. Then we count with respect to the number of disconnected horizontal and vertical wires. □

Corollary 1: *The information capacity of an* $N \times N$ *CSN is asymptotically less than* $2N \cdot \log N$ *bits as* $N \to \infty$.

Proof: Starting from equation (3) we have:

$$\begin{aligned} \boldsymbol{D} &= 1 + \sum_{q=1}^{N} \sum_{i=0}^{N-q} \sum_{j=0}^{N-q} \binom{N}{i}\binom{N}{j} S(N-i,q)S(N-j,q)q! \\ &< \sum_{q=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \binom{N}{i}\binom{N}{j} S(N-i,q)S(N-j,q)q! \end{aligned} \quad (4)$$

From (1) we have that:

$$S(N-i,q) = \frac{1}{q!}\sum_{k=0}^{q}(-1)^{q-k}\cdot\binom{q}{k}k^{N-i} \le \frac{1}{q!}\sum_{k=0}^{q}\binom{q}{k}k^{N-i} \quad (5)$$

similarly we have:

$$S(N-j,q) \le \frac{1}{q!}\sum_{r=0}^{q}\binom{q}{r}r^{N-j} \quad (6)$$

Replacing (5) and (6) into (4) we get:

$$\begin{aligned} \boldsymbol{D} &\le \sum_{q=1}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N}\binom{N}{i}\binom{N}{j}\frac{1}{q!}\sum_{k=0}^{q}\sum_{r=0}^{q}\binom{q}{k}\binom{q}{r}k^{N-i}r^{N-j} \\ &= \sum_{q=1}^{N}\frac{1}{q!}\sum_{k=0}^{q}\sum_{r=0}^{q}\binom{q}{k}\binom{q}{r}\sum_{i=0}^{N}\sum_{j=0}^{N}\binom{N}{i}\binom{N}{j}k^{N-i}r^{N-j} \\ &= \sum_{q=1}^{N}\frac{1}{q!}\sum_{k=0}^{q}\sum_{r=0}^{q}\binom{q}{k}\binom{q}{r}(1+k)^N(1+r)^N \\ &< \sum_{q=1}^{N}\frac{1}{q!}(1+q)^{2N}\sum_{k=0}^{q}\sum_{r=0}^{q}\binom{q}{k}\binom{q}{r} \\ &= \sum_{q=1}^{N}\frac{1}{q!}(1+q)^{2N}\cdot 2^{2q} \\ &< (1+N)^{2N}\cdot e^4 \end{aligned}$$

Therefore we have $\log\boldsymbol{D} < 2N\log(1+N) + 4\log(e)$ and the proof of the lemma is complete. □

4. CONCLUSIONS

The information storage capacity of crossbar switching networks, with regular contact switches, was derived explicitly. It was also shown that the capacity of an $N \times N$ network is asymptotically between $N\log N$ and $2N\log N$. Comparing the result to the information capacity of crossbar switching networks with semi-conducting switches, which is N^2, we conclude that the later ones, if available, are much more efficient as information storage devices.

5. REFERENCES

[1] T. Rueckes, K. Kim, E. Jose-levich, G. Y. Tseng, C.L. Cheung, and C.M. Lieber, *"Carbon nanotube based non-volatile random access memory for molecular computing"*, Science, No. 289, pp. 94–97, 2000.

[2] P.J. Kuekes, S. Williams and J.R. Heath, *"Molecular Wire Crossbar Memory"*, US Patent 6,128,214, Oct. 2000.

[3] J.R. Heath, P.J. Kuekes, G.S. Snider, R.S. Williams, *"A Defect-Tolerant Computer Architecture: Opportunities for Nanotechnology"*, Science, Vol. 280, June 1998.

[4] S.C. Goldstein, M. Budiu, *"NanoFabrics: spatial computing using molecular electronics"*, Int. Symp. on Computer Architecture, pp. 178-189, 2001.

[5] P.J. Kuekes, R.S. Williams, *"Demultiplexer for a molecular wire crossbar network"* (MWCN DEMUX), US Patent 6,256,767 B1, July 2001.

[6] P.J. Kuekes, R.S. Williams, J.R. Heath, *"Molecular-Wire Crossbar Interconnect (MWCI) for Signal Routing and Communications"*, US Patent 6,314,019 B1, Nov. 2001.

[7] A. DeHon, *"Array-based architecture for molecular electronics"*, 1st. workshop Non-silicon Comp. (NSC-1), 2002.

[8] M.M. Ziegler, M.R. Stan, *"Design and analysis of crossbar circuits for molecular nanoelectronics"*, IEEE Conf. on Nanotechnology, pp. 323 -327, 2002.

[9] P. P. Sotiriadis, *"On the Information Storage Capacity of Crossbar Switching Networks"*, Electronics Letters (subm).

[10] J.H. Van Lint and R.M. Wilson, *"A Course in Combinatorics"*, Cambridge University Press 1999.

[11] M. Abramowitz and I.A. Stegun, *"Handbook of Mathematical Functions"*, National Bureau of Standards, 6th edition, Nov. 1967