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A Fast Algorithm for the Eigenvalues Computation of a Toeplitz Matrix

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Abstract: This paper is concerned with the computation of the eigenvalues of a real, symmetric Toeplitz matrix. A very efficient algorithm is described which is based on the bisection method and on the Durbin algorithm.

Keywords: Algorithms theory, Eigenvalues computation, applied matrix theory.

I. INTRODUCTION

The problem of determining the eigenvalues and eigenvectors of a given matrix is, in general, one of exceptional difficulty. A large number of various algorithms, each applicable to a particular class of matrices, have been developed for this purpose. All these algorithms are based on the iteration of various procedures. Excellent surveys of the various algorithms can be found in [1] and [2].

A new idea for the computation of the minimum eigenvalue $\lambda_{min}(R)$ of a positevely definite Toeplitz matrix R is the following [3], [4].

Consider the matrix $R-\lambda I$. We have $\lambda<\lambda_{min}(R)$, if and only if $R-\lambda I>0$, as well as, $\lambda>\lambda_{min}(R)$, if and only if $R-\lambda I$- has at least one negative eigenvalue. Therefore, if one could find a criterion for the positivity of $R-\lambda I>0$, one could find a criterion for checking if $\lambda<\lambda_{min}(R)$. If the given matrix is Toeplitz, the criterion in question is the Durbin algorithm.

Suppose that $[A_0,B_0]$ is an a-priori known interval in which λ_{min} is lying, so $\lambda_{min} \in [A_0,B_0]$. The value of λ_{min} can be approximated by the value of λ using the following algorithm

Step 1 : $A:=A_0$, $B:=B_0$

Step ~~2 : $\lambda:=\frac{A+B}{2}$~~

Step 3 : If $R-\lambda I>0$ Then $A:=\lambda$, else $B:=\lambda$

Step 4 : If $|A-B|>2\varepsilon$,

(where ε is the absolute error),

Then go to Step 2

Step 5 : $\lambda:=\frac{A+B}{2}$

Step 6 : End

Note that after each iteration of steps 1 and 2, the interval which contains λ_{min} is subdivided and a final error estimation after the k-th iteration is

$$|\lambda-\lambda_{min}|\leq\varepsilon\leq\frac{Bo-Ao}{2^{k+1}}$$

The main advantage of the present algorithm is its sureconvergence and the absolute bound of the final error.

The paper is organised as follows. In Section II, some basic theoretical results from Matrix Theory are presented. In Section III, we combine these results in order to produce the proposed algorithm in its final form. In Section IV, the computational complexity of the algorithm is given. Finally, there exists a conclusion.

II. PRELIMINARY RESULTS

In this Section, some important Theorems from Matrix Theory are given. First, a definition is stated.

Definition 1: Leading $k\times k$ principal submatrix of a $n\times n$, $n\geq k$, matrix R is the $k\times k$ $R_k=[r_{ij}]$, $i,j=1..k$ ∎

The following proposition is proved in [5] (pp.103-104).

Proposition 1(Interlacing Property)

Let R_k be the leading $k\times k$ principal submatrix of the nxn symmetric matrix R, then $\forall$ k=1...n-1 one has:

$$\lambda_{k+1}(R_{k+1})\leq\lambda_k(R_k)\leq\lambda_k(R_{k+1})\leq$$

$$\leq\ldots\leq\lambda_2(R_{k+1})\leq\lambda_1(R_k)\leq\lambda_1(R_{k+1})$$

where $\lambda_1(R_{k+1}), \lambda_2(R_{k+1}), \ldots \lambda_{k+1}(R_{k+1})$

and

$\lambda_1(R_k), \lambda_2(R_k), \ldots, \lambda_{k-1}(R_k), \lambda_k(R_k)$.

are the eigenvalues of the matrices R_{k+1} and R_k respectively. It is assumed that

$\lambda_k(R_k)\leq\ldots\leq\lambda_2(R_k)\leq\lambda_1(R_k)$

and

$\lambda_{k+1}(R_{k+1})\leq\ldots\leq\lambda_2(R_{k+1})\leq\lambda_1(R_{k+1})$

Proof: See in [5], pp.103-104. ∎

Theorem 1

Let R be a symmetric $n\times n$ matrix and R_k be its leading $k\times k$ principal submatrix for k=1...n and $d_k=\det(R_k)$. If $d_k\neq0$ for all k=1...n, then the matrix R appears to have m-negative eigenvalues and (n-m) positive ones, where m is the number of sign alterations of the sequence $\{1,d_1,d_2,\ldots,d_n\}$.

Proof: Suppose that p of the k eigenvalues R_k are negative, that is

$$\lambda_k(R_k)\leq\lambda_{k-1}(R_k)\leq\ldots\leq$$
$$\leq\lambda_{k-p+1}(R_k)<0<\lambda_{k-p}(R_k)\leq\ldots\leq\lambda_1(R_k)$$

It is clear that

$$\mathrm{sign}(\det(R_k))=(-1)^p$$

According to Proposition 1, p eigenvalues of R_{k+1} are negative and (k-p) are positive. For the $\lambda_{k-p+1}(R_{k+1})$ it holds that:

$\lambda_{k-p+1}(R_k)\leq\lambda_{k-p+1}(R_{k+1})\leq\lambda_{k-p}(R_k)$,

$\lambda_{k-p+1}(R_{k+1})\neq0$ (according to our assumption).

One can verify that $\mathrm{sign}(\det(R_{k+1})) = (-1)^p\,\mathrm{sign}(\lambda_{k-p+1}(R_{k+1}))$ or $\mathrm{sign}(\det(R_{k+1})) = \mathrm{sign}(\det(R_k))\cdot\mathrm{sign}(\lambda_{k-p+1}(R_{k+1}))$. Therefore if there exists a sign alteration from d_k to d_{k+1}, then R_{k+1} has one more negative eigenvalue than R_k has. If d_k and d_{k+1} are of the same sign then R_k and R_{k+1} have the same number of negative eigenvalues. The proof is completed by induction. ∎

In the case that the matrix R is a symmetric Toeplitz matrix, Theorem 1 can be applied directly to the results found by the Durbin algorithm. Suppose now that $R=(r_{|i-j|})_{i,j=1...n}$

From Durbin's algorithm ([1]) one obtains $a_0=r_0$, $a_m=\frac{\det(R_{m+1})}{\det(R_m)}$, m=1,..,n-1 ,where R_m is the leading m×m principal submatrix of R [1] . Therefore $R_1=(r_0)$ and $\det(R_m)=a_{m-1}a_{m-2}\dots a_0$, m=1,..,n .

Now, we are interested in the number of sign alternation in the sequence $1,a_0,a_0a_1,\dots,a_0a_1\dots a_{n-1}$. This number is equal to the number of negative terms in the sequence $a_0,a_1,\dots,a_{n-1}$.(Note that from the assumption of the first theorem $a_m\neq 0$ m=0,..,n-1).

Definition: Let R be a symmetric n×n Toeplitz matrix. We define the function DN of R as follows: DN(R)=(The number of negative terms in the sequence $a_0,a_1,\dots,a_{n-1}$), where a_m,m=0,..,n-1, result by the application of Durbin's Algorithm on R. ∎

For the Durbin's algorithm, it is well known that a sign alternation exists from a_{m-1} to a_m, (i.e.$a_{m-1}a_m<0$) if and only if $|k_m|>1$,m=1,..,n-1 . For this reason, we can also define DN(R) as:

$$DN(R) \quad =(r_0<0)+\sum_{i=1}^{n-1}(1-|k_i|<0)$$

, where the operator (x<0) is defined as follows:

$$(x<0) = 1 \text{ if } x<0$$
$$(x<0) = 0 \text{ if } x\geq 0$$

One can easily seen, based on Theorem 1 and the above remarks that DN(R) = (The number of the negative eigenvalues of R).
In the following Proposition a new assisting parameter λ is introduced.

Proposition 2

Let R be a symmetric n×n Toeplitz matrix and $\lambda_r(R)\leq\lambda_{r-1}(R)\leq\dots\leq\lambda_1(R)$ .Then $\lambda=\lambda_m(R)$ if and only if $\forall\varepsilon>0$ ~~$DN(R-(\lambda+\varepsilon)I)>n-m$~~ and $DN(R-(\lambda-\varepsilon)I)\leq n-m$.

Proof: Since R is symmetric. there exists an orthonormal matrix Q such that $R=QDQ^T$. It can be assumed, without loss of generality, that $D=\mathrm{diag}(\lambda_1(R),\lambda_2(R),\dots,\lambda_n(R))$.

From the relation $vI = vQQ^T$ one takes

$$R-vI=Q(D-vI)Q^T=$$

$$= Q\,\mathrm{diag}(\lambda_1(R)-v,\ ,\lambda_2(R)-v,\dots,\lambda_m(R)-v,\dots,\lambda_n(R)-v)Q^T.$$

For $v=\lambda+\varepsilon=\lambda_m+\varepsilon$ we have

$\lambda_m(R)-v<0,\dots,\lambda_n(R)-v<0$

or equivalently $DN(R-(\lambda+\varepsilon)I)>n-m$,

for $v=\lambda-\varepsilon=\lambda_m-\varepsilon$ we have

$\lambda_1(R)-v>0,\dots,\lambda_m(R)-v>0$

or equivalently $DN(R-(\lambda-\varepsilon)I)\leq n-m$ ∎

In order to exploit Proposition 2, it now remains to estimate an initial interval containing the eigenvalues of a Toeplitz matrix. For this purpose, the following Theorem is proved in [6] (p.371).

Theorem 2: *(Gershgorin Circle Theorem)*

The eigenvalues $\lambda_1(R),\lambda_2(R),\dots,\lambda_n(R)$ of the matrix $R\in C^{n\times n}$ fulfill the relation

$$(\lambda_1(R),\lambda_2(R),\dots,\lambda_n(R))\subset$$

$$\subset\bigcup_{i=1}^{n}\left\{z\in C \quad :|z-r_{ii}|\leq\sum_{\substack{j=1\\ j\neq i}}^{n}|r_{ij}|\right\}$$ ∎

In the case where R is a symmetric Toeplitz matrix, Theorem 2 gives

$$A_0=$$

$$r_0-2\cdot\sum_{k=1}^{n-1}|r_k|=\lambda_n(R)\leq\lambda_{n-1}(R)\leq\dots\leq\lambda_1(R)\leq r_0+2\cdot\sum_{k=1}^{n-1}|r_k|=B_0$$

Now, having the initial interval $[A_0,B_0]$, and using the Proposition 2 one can formulate the following algorithm for finding the eigenvalues of the symmetric Toeplitz matrix.

III. THE ALGORITHM

In this section. the proposed algorithm is stated:

Algorithm:

Given the real symmetric $n \times n$ Toeplitz Matrix $R=(r_{|i-j|})_{i,j=1..n}$ or equivalent its elements, $r_0, r_1, \dots r_{n-1}$ and the small positive constant δ (for example $\delta = 0.001$),

Find the k-th eigenvalue of R, $\lambda_k(R)$, with respect to the order of the eigenvalues $\lambda_n(R) \le \lambda_{n-1}(R) \le \dots \le \lambda_1(R)$.

Error: If λ is an estimation of $\lambda_k(R)$, $\varepsilon = |\lambda - \lambda_k(R)|$ the absolute error, we demand $\varepsilon \le \varepsilon^*$, where ε^* is the defined bound of ε.

Step 1 : $A_0 := r_0 - 2 \cdot \sum_{k=1}^{n-1} |r_k|$, $B_0 := r_0 + 2 \cdot \sum_{k=1}^{n-1} |r_k|$

Step 2 : $A := A_0$, $B := B_0$

Step 3 : $S := \left\lceil \log_2 \left(\frac{Bo - Ao}{\varepsilon^*} \right) - 1 \right\rceil$

Step 4 : $\lambda := \frac{(1+d)A + (1-d)B}{2}$

Step 5 : For j=1:S

Step 6 : If $DN(R - \lambda I) \le n-k$ Then $A := \lambda$, else $B := \lambda$

Step 7 : $\lambda := \frac{A+B}{2}$

End

Remark 1: The symbol $\lceil a \rceil$ is used for the upper integer part of a number a. S is the number of recursions in the **for-loop** and results from the bound ε^*. One can easily verifies that $\varepsilon < \varepsilon^* \le \frac{Bo-Ao}{2^{S+1}}$

Remark 2: The small constant δ is essential: If we set $\delta = 0$, then the first value of L would be $L = r_0$ and the matrix $(R - r_0 I)$ would have its first leading subdeterminant equal to zero. In that case, an error "division by zero" would appeared in the call of the function DN. This constant increase the error bound ε by the negligible quantity $\delta \cdot \varepsilon$.

Remark 3: The substraction of λI from the matrix R, effects only its diagonal, so r_0 is replaced by $(r_0 - \lambda) I$.

IV. COMPUTATIONAL MULTIPLEXITY OF THE ALGORITHM

The Durbin recursion appears approximatel $2n^2$ MADs (Multiplications and Divisions) [1]. The number of required calls of function DN is S where

$$S := \left\lceil \log_2 \left(\frac{B_0 - A_0}{\varepsilon^*} \right) - 1 \right\rceil \le \log_2 \left(\frac{B_0 - A_0}{\varepsilon^*} \right) = \log_2 \left(\frac{4}{\varepsilon^*} \sum_{k=1}^{n-1} |r_k| \right)$$

So, the total cost in MADs (Multiplications and Divisions) is P, where

$$P = 2n^2 \log_2 \left(\frac{4}{\varepsilon^*} \sum_{k=1}^{n-1} |r_k| \right)$$

If the matrix R is an autocorrelation matrix and the relation $r_0 \ge |r_k|$, $\forall k \in (1, .., n-1)$ holds then $P \le 2n^2 \left[\log_2 (n-1) + \log_2 \left(\frac{4r_0}{\varepsilon^*} \right) \right]$.

The total multiplexity can be reduced more than 25% if one wants to compute all the eigenvalues of R.

V. CONCLUSION

A new fast and efficient algorithm for the computation of the eigenvalues of a symmetric, real, Toeplitz matrix is proposed. The algorithm has the following advantages:

- It converges always, since it is based on the bisection method.
- Multiple eigenvalues do not effect it.
- A bound for the absolute error is a-priori known
- Each eigenvalue can be computed directly, without computing all the absolute greater eigenvalues (as in the method of the Powers), and so no additive computational errors accumulate.
- The proposed algorithm can be developed in a parallel version using parallel Durbin or Shur recursions.

In Appendix, a Pascal Program is given.

APPENDIX

```
   { Const N=8; (dimension of R)
{   N1=7; (N1=n-1)
   { Type Vector=Array[0..N1] of Real; }
   {      EigenIndex=1..N; }
   { VAR U:VECTOR; }

    Function
Topeigen(r:Vector;K:EigenIndex;e:Real):Real;
{r=(r0,r1,...,rn-1,x)
{ K is the index of the eigenvalue}
{ e is the bound of the absolute error}

    Const D=0.001; ( D is the small constant δ
}
    Var A,B,L,H:Real;
        S,J:Integer;

    Function DN(L:Real):integer; {the function
DN}.
    Var Y,Z:Vector:
        K,A,r0:Real:
        M,I,SUM:Integer;
    Begin

      r0:=r[0];
      r[0]:=r0-L; {Durbin's algorithm}
      K:=-r[1]/r[0];
      A:=r[0];
      If A<0 then SUM:=1 Else SUM:=0;
      y[1]:=K;
      For M:=1 to N-2 do
      Begin
        a:=(1-K*K)*a;
        If A<0 then SUM:=SUM+1;
        K:=r[M+1];
        For I:=1 to M do
        K:=K+r[M-I+1]*Y[I];
        K:=-K/A;
        For I:=1 to M do
        Z[I]:=Y[I]+K*Y[M+1-I];
        For I:=1 to M do
        Y[I]:=Z[I];
        Y[M+1]:=K;
      End;
      a:=(1-K*K)*a:
      If A<0 then SUM:=SUM+1;
      r[0]:=r0;
      DN:=SUM:
    End;

    Begin ( main part of the algorithm)
      H:=0:
      For J:=1 to N-1 do
      H:=H+Abs(r[J]);
      A:=r[0]-2*H;
      B:=r[0]+2*H;
      S:=Round(ln((B-A)/e)/ln(2)-0.5);
      L:=((1+D)*A+(1-D)*B)/2;
      For J:=1 to S do
      Begin
        If DN(L)<=N-K then A:=L Else B:=L;
        L:=(A+B)/2;
      end:
      Topeigen:=L:
    End: {Topeigen}

BEGIN
  U[0]:=1;
  U[1]:=-50;
  U[2]:=0;
  U[3]:=1;
  U[4]:=7;
  U[5]:=43;
  U[6]:=9:
  U[7]:=0:
  WRITELN:
  WRITELN(TOPEIGEN(U,8,1e-10):4:10):


  {R=(u|i-j|)i,j=1,..,8}    (5th
eigenvalue)
{e=1E-10}
    END.
```

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